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A COMPARISON OF DIGITAL COMPUTER PROGRAMS  
FOR THE NUMERICAL SOLUTION OF ORDINARY  
DIFFERENTIAL EQUATIONS


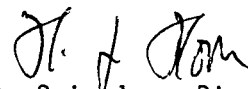
By Hugo L. Ingram  
Aero-Astrodynamics Laboratory

July 1, 1973

**NASA**

*George C. Marshall Space Flight Center  
Marshall Space Flight Center, Alabama*



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| 16. ABSTRACT<br><p>Recently the determination of the best technique for numerically solving systems of ordinary differential equations on a digital computer has received much attention. Phyllis Fox in A Comparative Study of Computer Programs for Integrating Differential Equations; and Hull, Enright, Fellen, and Sedgewick in Comparing Numerical Methods for Ordinary Differential Equations made studies on the computational efficiency of several different numerical integration techniques, but their studies did not include the Runge-Kutta formulas developed by Fehlberg (NASA TR R-287 and R-315). The use of these formulas in conjunction with a stepsize control developed in this report is explained, and one of the formulas is chosen for comparison with other integration techniques. This comparison of one of the best of Fehlberg's formulas with the different numerical techniques described in the aforementioned studies on a variety of test problems clearly shows the superiority of Fehlberg's formula. That is, on each of the test problems, the chosen Fehlberg formula is able to achieve a given accuracy in less computer time than any of the other techniques tested. Also, the computer program for the chosen Fehlberg formula is less complex and easier to use than the computer programs for most of the other techniques. To illustrate the use of the chosen Fehlberg formula, a computer listing of its application to several example problems is included along with a sample of the computer output from these applications.</p> |  |  |  |  |
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## A COMPARISON OF DIGITAL COMPUTER PROGRAMS FOR THE NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS

When a physical problem is to be simulated with the aid of a digital computer, the result is often a system of ordinary differential equations that must be solved numerically. For this reason an efficient numerical integration algorithm is very desirable to minimize the amount of computer time needed to solve a particular problem.

Many different approaches are available for the numerical solution of system of ordinary differential equations, and an evaluation in this report of some of these approaches was motivated by a desire to compute optimal trajectories as rapidly as possible. In computing optimal trajectories, the system of ordinary differential equations known as the equations of motion and adjoint variables must be integrated many times to satisfy iteratively the boundary conditions. This process consumes much computer time unless the particular numerical integration technique being used is operating efficiently. The Runge-Kutta formulas derived by E. H. Fehlberg in References 1 and 2 and discussed in the next sections of this report meet this requirement of operational efficiency better than any of the other methods tested.

Also described in the section of this report entitled An Explanation of Runge-Kutta Numerical Integration Formulas is a new stepsize control procedure for Fehlberg's numerical integration formula. This new stepsize control procedure is combined with one of the best of Fehlberg's formulas, and the results it achieves for a variety of test problems is compared with the results achieved by other numerical integration techniques for the same test problems. The data obtained from these comparisons is discussed in detail in the sections of this report entitled Fehlberg's Runge-Kutta Formulas With Stepsize Control and A Comparison of Fehlberg's 7-8-13 Formulas With the Numerical Integration Techniques of References 3 and 4.

# AN EXPLANATION OF RUNGE-KUTTA NUMERICAL INTEGRATION FORMULAS

Consider the system of differential equations denoted by

$$\dot{x} = f(t, x) \quad , \quad (1)$$

where  $\dot{x}$ ,  $f$ , and  $x$  are all vectors of dimension  $n$ . Note that the symbol  $\dot{x}$  denotes  $dx/dt$  where  $t$  is assumed to be the independent variable for the system. For a given set of initial conditions denoted by  $x(t_0) = x_0$  values for  $x(t_0 + \Delta t)$  can be obtained from Runge-Kutta formulas as follows:

$$x(t_0 + \Delta t) = x_0 + \Delta t \sum_{k=0}^m c_k f_k \quad , \quad (2)$$

where

$$\begin{aligned} f_0 &= f(t_0, x_0) \\ f_k &= f(t_0 + \alpha_k \Delta t, x_0 + \Delta t \sum_{\ell=0}^{k-1} \beta_{k\ell} f_\ell) \text{ for } k = 1, 2, \dots, m \end{aligned} \quad . \quad (3)$$

The coefficients  $\alpha_k$ ,  $c_k$ ,  $\beta_{k\ell}$ , and the integer  $m$  are determined to make the expression for  $x(t_0 + \Delta t)$  as given in equation (2) equal to a Taylor series expansion for  $x(t_0 + \Delta t)$  up to a certain order. For this determination of the coefficients to be meaningful, the vector function  $f(t, x)$  must be reasonable enough to have a convergent Taylor series expansion in some neighborhood of the initial conditions  $t_0, x_0$ .

As an example of how the coefficients  $\alpha_k$ ,  $c_k$ , and  $\beta_{k\ell}$  are determined, a second-order Runge-Kutta formula can be derived. To do this, consider a Taylor series expansion of  $x(t_0 + \Delta t)$  to second order. That is,



$$x(t_o + \Delta t) = x_o + \dot{x}_o \Delta t + \frac{1}{2} \ddot{x}_o \Delta t^2 \quad (4)$$

Then note that

$$\dot{x}_o = f(t_o, x_o)$$

and

$$\begin{aligned} \ddot{x}_o &= \left[ \frac{d}{dt} (\dot{x}) \right]_{x=x_o} = \left[ \left( \frac{\partial f}{\partial t} \right) + \left( \frac{\partial f}{\partial x} \right) \dot{x} \right]_{x=x_o} \\ &= \left[ \left( \frac{\partial f}{\partial t} \right) + \left( \frac{\partial f}{\partial x} \right) f(t, x) \right]_{x=x_o} \\ &= \left( \frac{\partial f}{\partial t} \right)_{x=x_o} + \left( \frac{\partial f}{\partial x} \right)_{x=x_o} f(t_o, x_o) \end{aligned}$$

Thus the Taylor series expansion can be rewritten as:

$$x(t_o + \Delta t) = x_o + f(t_o, x_o) \Delta t + \frac{1}{2} \left[ \left( \frac{\partial f}{\partial t} \right)_o + \left( \frac{\partial f}{\partial x} \right)_o f(t_o, x_o) \right] \Delta t^2 \quad (5)$$

Now the expression for  $x(t_o + \Delta t)$  from the Runge-Kutta approach is given by equation (2). That is,

$$x(t_o + \Delta t) = x_o + \Delta t [c_o f(t_o, x_o) + c_1 f_1 + \dots + c_m f_m] \quad (6)$$

Now equation (3) can be used and  $f_1$  expanded in a multivariable Taylor series to give:

$$\begin{aligned}
f_1 &= f(t_0 + \alpha_1 \Delta t, x_0 + \Delta t \beta_{10} f_0) \\
&= f(t_0, x_0) + \left(\frac{\partial f}{\partial t}\right)_0 (\alpha_1 \Delta t) + \left(\frac{\partial f}{\partial x}\right)_0 [\Delta t \beta_{10} f_0] + \dots \quad (7)
\end{aligned}$$

When the above expression is substituted into equation (6), second-order terms will be obtained in  $\Delta t$ . Thus, there is no need to carry the expansion of equation (7) past first order and  $m$  in equation (6) is chosen to be one. Thus equation (6) becomes:

$$\begin{aligned}
x(t_0 + \Delta t) &= x_0 + \Delta t \left\{ c_0 f(t_0, x_0) \right. \\
&\quad \left. + c_1 \left[ f(t_0, x_0) + \left(\frac{\partial f}{\partial t}\right)_0 (\alpha_1 \Delta t) + \left(\frac{\partial f}{\partial x}\right)_0 (\Delta t \beta_{10} f_0) \right] \right\} \quad (8)
\end{aligned}$$

Therefore,

$$\begin{aligned}
x(t_0 + \Delta t) &= x_0 + (c_0 + c_1) f(t_0, x_0) \Delta t \\
&\quad + \left[ \left(\frac{\partial f}{\partial t}\right)_0 (c_1 \alpha_1) + \left(\frac{\partial f}{\partial x}\right)_0 f(t_0, x_0) c_1 \beta_{10} \right] \Delta t^2 \quad (9)
\end{aligned}$$

Now a comparison of equations (5) and (9) shows that:

$$\begin{aligned}
c_0 + c_1 &= 1 \\
c_1 \alpha_1 &= 1/2 \\
c_1 \beta_{10} &= 1/2 \quad . \quad (10)
\end{aligned}$$

Any set of coefficients that satisfies equation (10) will give a second-order Runge-Kutta formula that uses only two evaluations of the differential equations. Thus it can be seen that Runge-Kutta formulas are not unique. As an example  $c_0 = c_1 = 1/2$ ,  $\alpha_1 = 1$ , and  $\beta_{10} = 1$  will satisfy equation (10). Equation (10) is called the equations of condition for a second-order Runge-Kutta formula. Also, the number of evaluations of the differential equations required for a particular order is usually called the number of function evaluations needed. With this background information, the Runge-Kutta formulas developed by Fehlberg in References 1 and 2 can now be discussed.

## FEHLBERG'S RUNGE-KUTTA FORMULAS WITH STEP SIZE CONTROL

To obtain a useful stepsize control procedure for Runge-Kutta formulas some indication of the truncation error of the series expansion must be determined. Fehlberg's idea is to develop Runge-Kutta formulas of adjacent order that use the same function evaluations, and then the difference in the two formulas is a good approximation to some single term in the Taylor series expansion.

In References 1 and 2, Fehlberg performs the very difficult determination of adjacent Runge-Kutta formulas for orders from one to eight. That is, formulas of order 1 and 2, 2 and 3, 3 and 4, 4 and 5, 5 and 6, 6 and 7, 7 and 8, and 8 and 9 are all developed, and each adjacent pair uses the same function evaluations so that their difference can approximate a corresponding term in the Taylor series expansion for  $x(t_o + \Delta t)$ . As an example, for  $m=12$  a seventh-order Runge-Kutta formula and an eighth-order Runge-Kutta formula are developed that use the same function evaluations. That is,

$$x(t_o + \Delta t) = x_o + \Delta t \sum_{k=0}^{12} c_k f_k \quad (11)$$

$$\hat{x}(t_o + \Delta t) = x_o + \Delta t \sum_{k=0}^{12} \hat{c}_k f_k \quad (12)$$

where

$$f_0 = f(t_0, x_0)$$

$$f_k = f(t_0 + \alpha_k \Delta t, x_0 + \Delta t \sum_{\ell=0}^{k-1} \beta_{k\ell} f_\ell) \text{ for } k = 1, 2, \dots, 12 \quad . \quad (13)$$

Table 1, which is taken from Reference 1, gives the values for  $c_k$ ,  $\hat{c}_k$ ,  $\alpha_k$ , and  $\beta_{k\ell}$ . Equation (11) is a Runge-Kutta formula that agrees with the Taylor series expansion to order seven, and equation (12) is a Runge-Kutta formula that agrees with the Taylor series expansion to order eight. Thus, the difference between the two formulas that Fehlberg denotes by TE is a good approximation to the eighth-order term in the Taylor series expansion. This difference is given by the following expression.

$$TE = \frac{41}{840} (f_0 + f_{10} - f_{11} - f_{12}) \Delta t \quad . \quad (14)$$

Since the expression for TE as given by equation (14) is a good approximation to the eighth-order term in the Taylor series expansion for  $x(t_0 + \Delta t)$ , it can be used to determine  $\Delta t$ . Let  $|TE|$  denote the largest component of the vector TE as given in equation (14). Then

$$|TE| \approx a_8 \Delta t^8 \quad , \quad (15)$$

where  $a_8$  is the coefficient of the eighth-order term in the Taylor series expansion for the variable which yields the largest component of TE. Therefore,

$$a_8 \approx \frac{|TE|}{\Delta t^8} \quad . \quad (16)$$

TABLE 1. RK 7(8)

| $\alpha_k$       |                | $\beta_{kf}$         |                |                  |                    |                     |                   |                     |                 |                  |                 |    | $c_k$ | $\hat{c}_k$      |                  |
|------------------|----------------|----------------------|----------------|------------------|--------------------|---------------------|-------------------|---------------------|-----------------|------------------|-----------------|----|-------|------------------|------------------|
| $k \backslash f$ |                | 0                    | 1              | 2                | 3                  | 4                   | 5                 | 6                   | 7               | 8                | 9               | 10 | 11    |                  |                  |
| k                |                |                      |                |                  |                    |                     |                   |                     |                 |                  |                 |    |       |                  |                  |
| 0                | 0              | 0                    |                |                  |                    |                     |                   |                     |                 |                  |                 |    |       | $\frac{41}{340}$ | 0                |
| 1                | $\frac{2}{27}$ | $\frac{2}{27}$       |                |                  |                    |                     |                   |                     |                 |                  |                 |    |       | 0                | 0                |
| 2                | $\frac{1}{9}$  | $\frac{1}{36}$       | $\frac{1}{12}$ |                  |                    |                     |                   |                     |                 |                  |                 |    |       | 0                | 0                |
| 3                | $\frac{1}{6}$  | $\frac{1}{24}$       | 0              | $\frac{1}{8}$    |                    |                     |                   |                     |                 |                  |                 |    |       | 0                | 0                |
| 4                | $\frac{5}{12}$ | $\frac{5}{12}$       | 0              | $-\frac{25}{16}$ | $\frac{25}{16}$    |                     |                   |                     |                 |                  |                 |    |       | 0                | 0                |
| 5                | $\frac{1}{2}$  | $\frac{1}{20}$       | 0              | 0                | $\frac{1}{4}$      | $\frac{1}{5}$       |                   |                     |                 |                  |                 |    |       | $\frac{34}{105}$ | $\frac{34}{105}$ |
| 6                | $\frac{5}{6}$  | $-\frac{25}{108}$    | 0              | 0                | $\frac{125}{108}$  | $-\frac{65}{27}$    | $\frac{125}{54}$  |                     |                 |                  |                 |    |       | $\frac{9}{35}$   | $\frac{9}{35}$   |
| 7                | $\frac{1}{6}$  | $\frac{31}{300}$     | 0              | 0                | 0                  | $\frac{61}{225}$    | $-\frac{2}{9}$    | $\frac{13}{900}$    |                 |                  |                 |    |       | $\frac{9}{35}$   | $\frac{9}{35}$   |
| 8                | $\frac{2}{3}$  | 2                    | 0              | 0                | $-\frac{53}{6}$    | $\frac{704}{45}$    | $-\frac{107}{9}$  | $\frac{67}{90}$     | 3               |                  |                 |    |       | $\frac{9}{280}$  | $\frac{9}{280}$  |
| 9                | $\frac{1}{3}$  | $-\frac{91}{108}$    | 0              | 0                | $\frac{23}{108}$   | $-\frac{976}{135}$  | $\frac{311}{54}$  | $-\frac{19}{60}$    | $\frac{17}{6}$  | $-\frac{1}{12}$  |                 |    |       | $\frac{9}{280}$  | $\frac{9}{280}$  |
| 10               | 1              | $\frac{2383}{4100}$  | 0              | 0                | $-\frac{341}{164}$ | $\frac{4496}{1025}$ | $-\frac{301}{82}$ | $\frac{2133}{4100}$ | $\frac{45}{82}$ | $\frac{45}{164}$ | $\frac{18}{41}$ |    |       | $\frac{41}{840}$ | 0                |
| 11               | 0              | $\frac{3}{205}$      | 0              | 0                | 0                  | 0                   | $-\frac{6}{41}$   | $-\frac{3}{205}$    | $-\frac{3}{41}$ | $\frac{3}{41}$   | $\frac{6}{41}$  | 0  |       |                  | $\frac{41}{840}$ |
| 12               | 1              | $-\frac{1777}{4100}$ | 0              | 0                | $-\frac{341}{164}$ | $\frac{4496}{1025}$ | $-\frac{289}{82}$ | $\frac{2193}{4100}$ | $\frac{51}{82}$ | $\frac{33}{164}$ | $\frac{12}{41}$ | 0  | 1     |                  | $\frac{41}{840}$ |

Thus, for a given value of  $\Delta t$ , equation (14) can be used to compute TE. Then equation (16) can be used to compute  $a_8$ . Let  $\epsilon$  denote the largest acceptable component of TE. Then, if  $|TE| \leq \epsilon$ , the step taken with  $\Delta t$  is an acceptable step and a  $\Delta t$  for the next step can be computed. If  $|TE| > \epsilon$ , then the step taken with  $\Delta t$  is not acceptable, and the step must be taken over again with a new  $\Delta t$  that can be computed. Let  $\Delta t_n$  denote the new  $\Delta t$  that is to be computed and  $\Delta t_o$  denote the old  $\Delta t$ . To derive an expression for  $\Delta t_n$ , it is required that

$$\epsilon = a_8 \Delta t_n^8 \quad . \quad (17)$$

That is, the new  $\Delta t$  should produce a maximum component of the truncation error of magnitude  $\epsilon$ . Thus

$$\Delta t_n = \left( \frac{\epsilon}{a_8} \right)^{1/8} \quad . \quad (18)$$

Now the value of  $a_8$  is obtained from equation (16) with the value of  $|TE|$  produced by  $\Delta t_o$ . Thus,

$$\Delta t_n = \left[ \frac{\epsilon}{\left( \frac{|TE|}{\Delta t_o^8} \right)} \right]^{1/8} = \Delta t_o \left( \frac{\epsilon}{|TE|} \right)^{1/8} \quad (19)$$

Equation (19) gives the computed value of  $\Delta t_n$  that is used for the next integration step if  $|TE| \leq \epsilon$  and is used for a repeat step if  $|TE| > \epsilon$ . In actual practice, only 80 percent of the  $\Delta t_n$  as computed by equation (19) is used for either the next step or a repeat step. Also, in computing  $|TE|$  each component is usually divided by the corresponding component from the initial value of  $x$  so

that a valid comparison can be made in search of the maximum component. Several example computer listings are shown in Appendix A so that the different implementations of the normalization of the truncation error vector can be illustrated.

Note that in the example computer program listings, equation (12) is used to compute  $x(t_0 + \Delta t)$ . This allows more accuracy in the final result to be obtained than if equation (11) were used because equation (12) is an eighth-order formula. In fact, for most problems,  $\epsilon$  will be a bound for the entire solution time interval because of the conservative stepsize control procedure used. Other less conservative procedures could be implemented if computer speed is of more importance than accuracy of the solutions obtained, but the results in the next sections indicate that this conservative stepsize control gives very satisfactory computer execution times on the example problems. The formula chosen for testing on the example problem is the 7-8 formula which used 13 function evaluations (denoted by Fehlberg's 7-8-13 formula). Fehlberg states in Reference 2 that this formula is probably the best for difficult problems, but for easier problems where values of  $x(t)$  are needed at more frequent intervals, a lower order formula might be more efficient. If a lower order formula is used (for example, a 2-3 formula), then equation (19) can still be used to compute  $\Delta t$ , but the power  $(1/8)$  is replaced by  $(1/3)$ .

## A COMPARISON OF FEHLBERG'S 7-8-13 FORMULA WITH THE NUMERICAL INTEGRATION TECHNIQUES OF REFERENCES 3 AND 4

From Reference 3, two example problems are selected. These examples are denoted by B1 and F1. The definition of problem B1 is given as follows:

### Test Problem B1

$$\begin{array}{lll} \dot{x}_1 = x_1^2 x_2 & x_1(t_0) = 1 & t_0 = 0 \\ \dot{x}_2 = -1/x_1 & x_2(t_0) = 1 & t_f = 4 \end{array}$$

The definition of problem F1 is given as follows:

Test Problem F1

$$\dot{x}_1 = x_3$$

$$\dot{x}_2 = x_4$$

$$\dot{x}_3 = x_1 + 2x_4 - (1 - \mu) \frac{(x_1 + \mu)}{\left[(x_1 + \mu)^2 + x_2^2\right]^{3/2}} - (\mu) \frac{(x_1 - 1 + \mu)}{\left[(x_1 - 1 + \mu)^2 + x_2^2\right]^{3/2}}$$

$$\dot{x}_4 = x_2 - 2x_3 - (1 - \mu) \frac{x_2}{\left[(x_1 + \mu)^2 + x_2^2\right]^{3/2}} - (\mu) \frac{x_2}{\left[(x_1 - 1 + \mu)^2 + x_2^2\right]^{3/2}}$$

$$x_1(t_0) = 0.994$$

$$\mu = 0.012277471$$

$$x_2(t_0) = 0.$$

$$t_0 = 0$$

$$x_3(t_0) = 0.$$

$$t_f = 11.124340337$$

$$x_4(t_0) = -2.0317326296$$

Figure 1 shows the results obtained for test problem B1 using Fehlberg's 7-8-13 formula (denoted by RK713). Figure 2 shows the results obtained for test problem F1 using RK713. In Figure 1, the error shown is calculated as  $\epsilon = |e^4 - x(1)|$ . Also, note that the percentage error in  $x(2)$  is equal to the percentage error in  $x(1)$  because of the stepsize control for RK713 that was used for this problem. This can be seen by examining the computer program listing shown in Appendix A and the associated output for this problem.

Figure 5 of Reference 3 is comparable to Figure 1 of this report. That is, Figure 5 of Reference 3 shows the same type of information for the integration techniques tested in Reference 3, that is shown by Figure 1 of this report for RK713. From Figure 5 of Reference 3, it can be seen that a predictor-corrector technique (denoted by HPCG) uses the fewest function evaluations



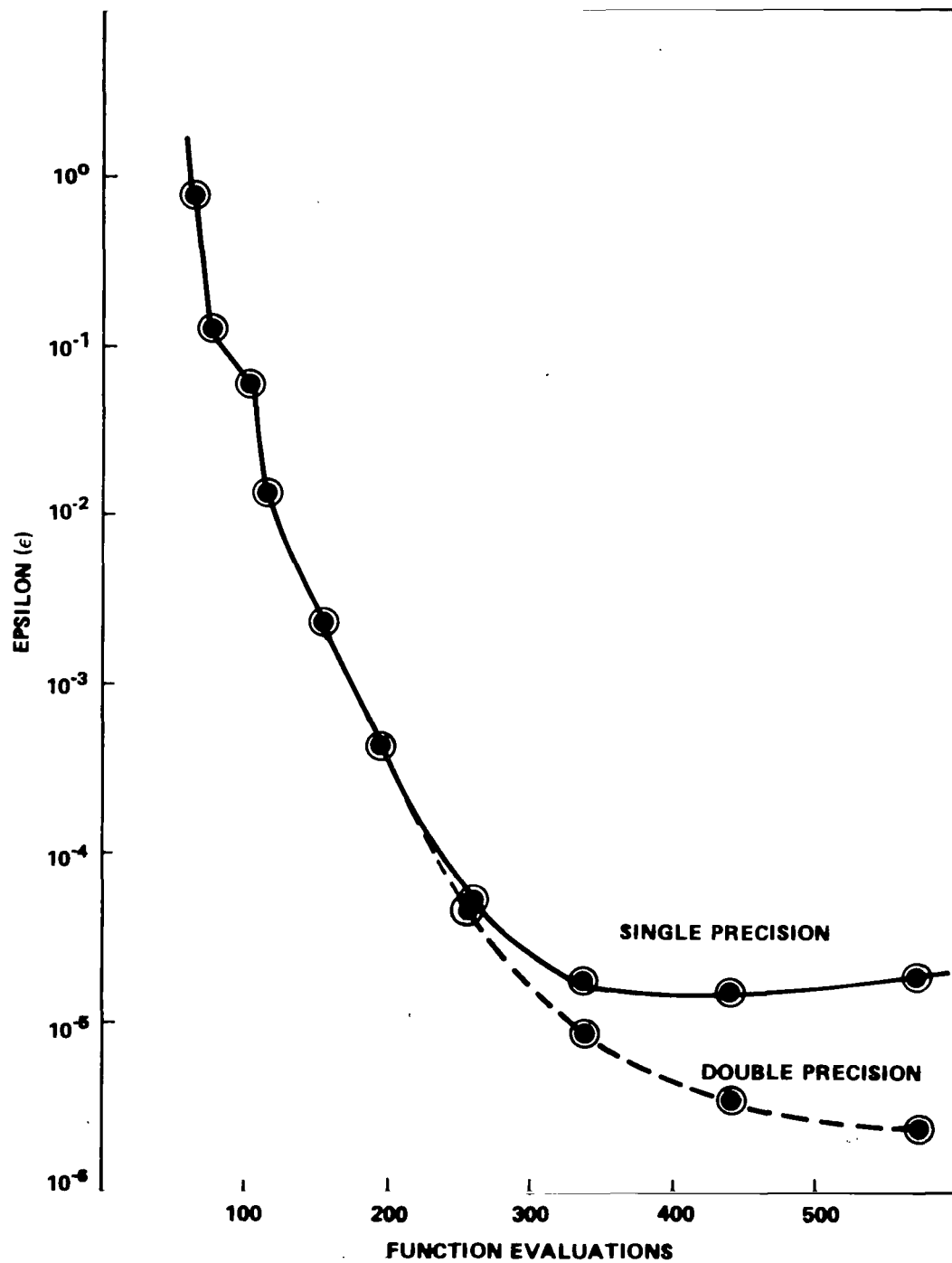


Figure 1. Test problem B1.

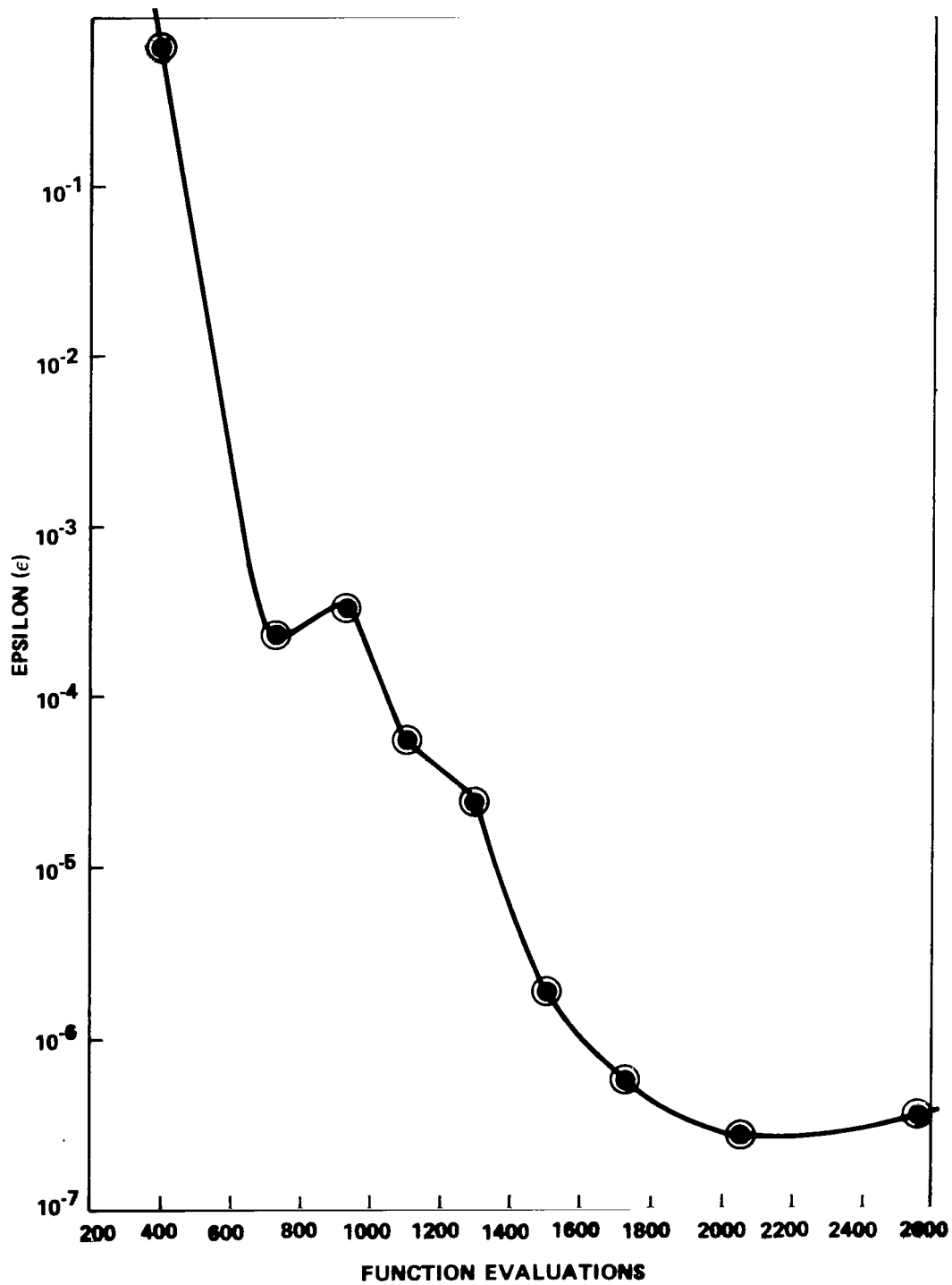


Figure 2. Test problem F1.

(about 100) for an  $\epsilon$  of  $10^{-1}$ . As seen in Figure 1, RK713 needs about 85 function evaluations for this accuracy. For an accuracy of  $\epsilon = 10^{-2}$ , RK713 needs only 120 function evaluations, whereas the only two integration techniques tested in Reference 3 that were able to achieve this accuracy (rational extrapolation and extrapolated Runge-Kutta) needed about 300 function evaluations. As seen in Figure 1, RK713 is able to achieve an  $\epsilon$  of  $10^{-4}$  with about 240 function evaluations. None of the integration techniques tested in Reference 3 was able to achieve this accuracy. The fact that the curve levels off at around  $10^{-5}$  accuracy is due to the precision limits of the computer used and not the difficulty of the problem or any inadequacy in the RK713 technique. The computer runs in this section were made on an XDS-930 computer unless otherwise indicated. This computer is able to carry only 11 or 12 significant figures for each computation. The dotted line in Figure 1 shows the results from the UNIVAC 1108 computer using double precision, which allows 16 to 18 significant figures to be considered in the computations.

When Figure 2 of this report is compared with Figure 6 of Reference 3, the superiority of RK713 is again demonstrated. The error  $\epsilon$  shown in Figure 2 of this report and Figure 6 of Reference 3 is computed by the formula

$$\epsilon = \left\{ [x_1(t_f) - x_1(t_o)]^2 + [x_2(t_f)]^2 \right\}^{1/2} .$$

RK713 is able to attain an  $\epsilon = 10^{-6}$  with 1600 function evaluations. Most of the integration techniques tested in Reference 1 could not achieve an accuracy of even  $\epsilon = 10^{-5}$ . The only two routines to come close were again the extrapolated Runge-Kutta routine and the rational extrapolation routine, and both required about 2500 function evaluations to get close to the accuracy  $\epsilon = 10^{-5}$ . Again a sample computer program listing and its output are included in Appendix A for this problem.

From Reference 4, two example problems were also selected for comparison purposes and denoted by the symbols B12 and E22. The definition of problem B12 is given as follows:

Test Problem B12 (this is also problem E of Reference 3)

$$\begin{array}{lll} \dot{x}_1 = 2(x_1 - x_1 x_2) & x_1(t_o) = 1 & t_o = 0 \\ \dot{x}_2 = -x_2(1 - x_1) & x_2(t_o) = 3 & t_f = 20 \end{array}$$

The definition of problem E22 is given as follows:

Test Problem E-22

$$\begin{aligned} \dot{x}_1 &= x_2 & x_1(t_0) &= 2 & t_0 &= 0 \\ \dot{x}_2 &= (1 - x_1^2) x_2 - x_1 & x_2(t_0) &= 0 & t_f &= 20 \end{aligned}$$

Figures 3 and 4 are plots of the errors achieved by RK713 versus the number of function evaluations needed for problems B12 and E22, respectively. To attempt to compare with the results shown in Reference 4, the error  $\epsilon$  shown in Figures 3 and 4 is computed as follows:

$$\epsilon = \frac{\max. [y_1(t_f) - x_1(t_f), y_2(t_f) - x_2(t_f)]}{t_f - t_0}$$

where  $[y_1(t_f), y_2(t_f)]$  are assumed to be the actual solution of the problems and are obtained by a very accurate integration of the two example sets of differential equations on the UNIVAC 1108 using double precision arithmetic.

From Figure 3 it can be seen that RK713 on problem B12 needs 600 function evaluations to achieve an accuracy of  $\epsilon = 10^{-3}$ , 915 function evaluations to achieve an accuracy of  $\epsilon = 10^{-6}$ , and 1615 function evaluations to achieve an accuracy of  $\epsilon = 10^{-9}$ . From Reference 4, the best method tested (with respect to the actual machine time needed to solve problem B12) is the Bulirsch-Stoer method. The Bulirsch-Stoer method needed 992 function evaluations for  $\epsilon = 10^{-3}$ , 1873 function evaluations for  $\epsilon = 10^{-6}$ , and 3128 function evaluations for  $\epsilon = 10^{-9}$ . Since the computer program DETEST mentioned in Reference 4 was not available, the other statistics given in Reference 4 were not compared. It is hoped that the authors of Reference 4 will be interested enough in RK713 to try it on their DETEST program.

From Figure 4, it can be seen that RK713 on problem E22 needs 665 function evaluations to achieve an accuracy of  $\epsilon = 10^{-3}$ , 915 function evaluations to achieve an accuracy of  $\epsilon = 10^{-6}$ , and 1625 function evaluations to achieve an accuracy of  $\epsilon = 10^{-9}$ . Again, the best method tested in Reference 4 (with respect to the actual machine time needed to solve problem E22) was the Bulirsch-Stoer method. On this problem, the Bulirsch-Stoer method needed 1109 function evaluations for  $\epsilon = 10^{-3}$ , 2009 function evaluations for  $\epsilon = 10^{-6}$ ,

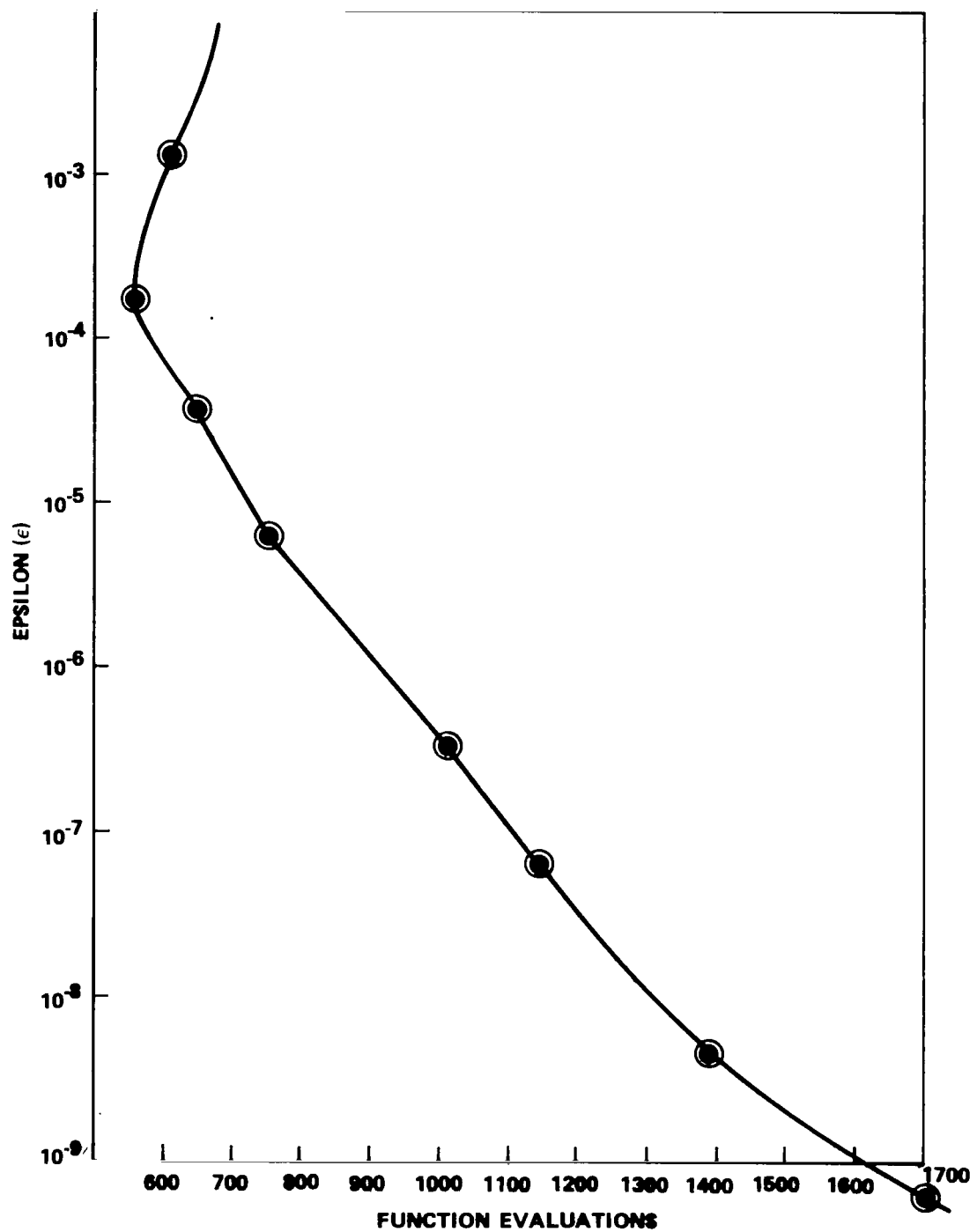


Figure 3. Test problem B12.

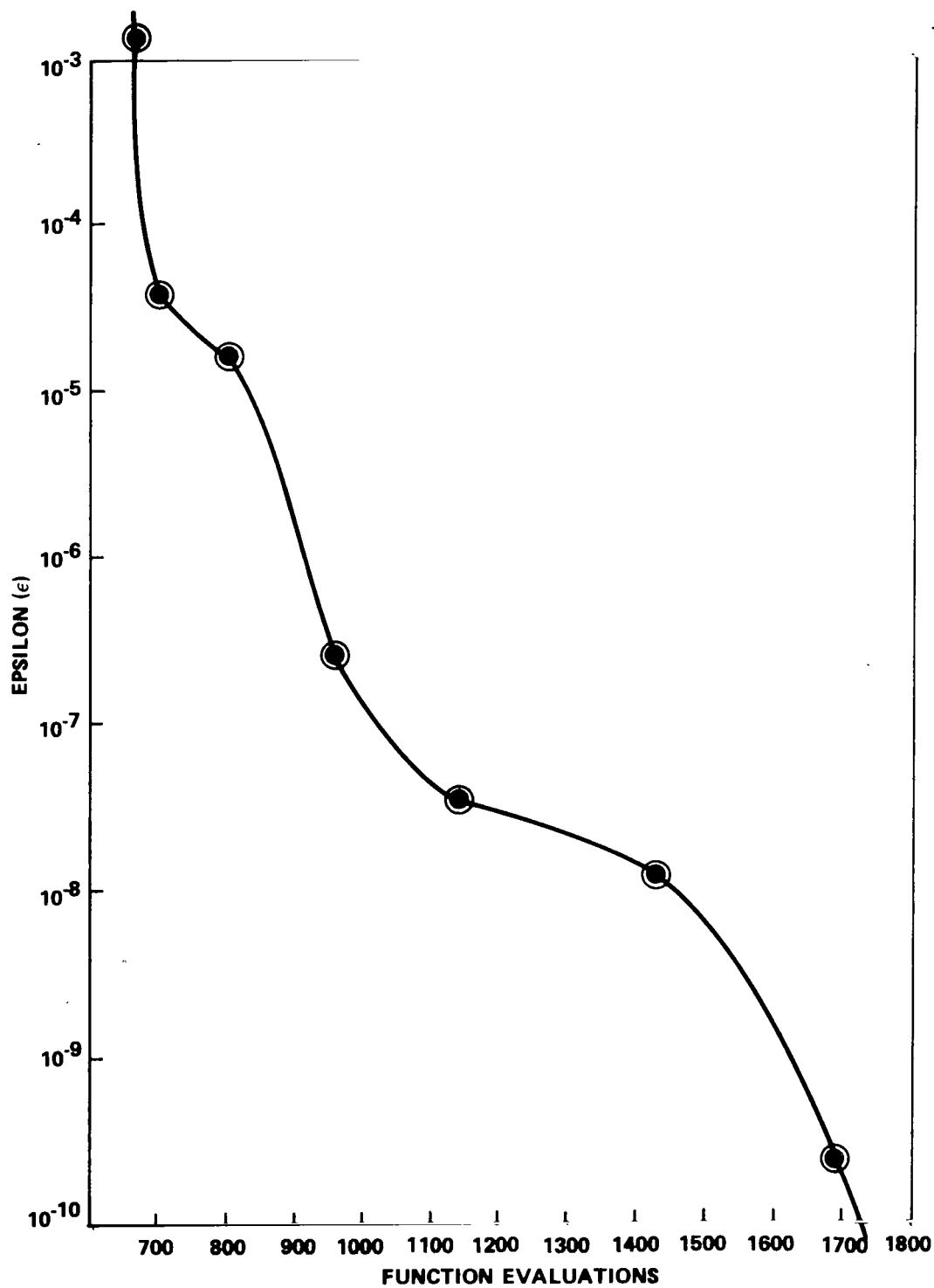


Figure 4. Test problem E22.

and 3588 function evaluations for  $\epsilon = 10^{-9}$ . Note again that a sample computer program listing and its output for both of the examples from Reference 4 are also contained in Appendix A. For each of the example problems of this section, the last output from each example computer program listing shown in Appendix A is a detailed print so that the actual behavior of each function can be seen. This detailed print was integrated as accurately as possible on the XDS 930 computer and also serves to illustrate the detailed print feature of the computer program.

All of the results of this section indicate that RK713 is superior to any of the other methods tested in References 3 and 4. Since all the errors in this section were absolute errors from the actual solutions of each of the differential equations, a more meaningful comparison is attempted in the next section. That is, RK713 and a rational extrapolation algorithm DIFSYP are compared on the basis of the accuracy each integration algorithm thinks it is achieving. Since actual accurate solutions of most differential equations attempted are not available, this is really the only comparison that can be made for these problems. Another reason for the inclusion of the comparison of the rational extrapolation algorithm with RK713 is that References 3 and 4 both concluded that the rational extrapolation algorithm as developed by Bulirsch-Stoer in Reference 5 was probably the best all-around algorithm they tested.

## MORE DETAILED COMPARISONS OF FEHLBERG'S 7-8-13 FORMULA WITH AN EXTRAPOLATION ALGORITHM BASED ON RATIONAL FUNCTIONS

In this section, a comparison of the two integration techniques (Fehlberg's 7-8-13 formula RK713 and the rational function extrapolation algorithm DIFSYP) is made using systems of differential equations describing the following example problems: planar elliptic vacuum orbits with eccentricities ranging from 0.001 to 0.991, a three-dimensional almost circular vacuum orbit, and two reentry trajectories with aerodynamic forces and the adjoint differential equations included.<sup>1</sup>

After preparation of this report was complete the author was informed that a newer version of DIFSYP had been developed by R. Bulirsch. This new

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1. The comparisons shown were performed by Dr. E. D. Dickmanns while he was working at Marshall Space Flight Center. They are included in this report with his permission. Dr. Dickmanns is now working in West Germany at the DFVLR - Institut für Dynamik der Flugsysteme.

version is claimed to be 30 to 40 percent faster than the old version for some accuracy requirements. Appendix B contains a listing of both the old and new versions of DIFSYP. No comparisons were made with the new version of DIFSYP because the results shown in this section and the previous section indicate that a 30- to 40-percent improvement in DIFSYP would not make it superior to RK713 for most accuracy requirements.

The following sections describe in detail the example problems and the comparisons that were made between RK713 and the old version of DIFSYP.

## Systems of Differential Equations Used

Planar Elliptic Orbits. For the range angle from pericenter  $\theta$  (eccentric anomaly) as independent variable the equations of motion are

$$\frac{du}{d\theta} = -\dot{r}$$

$u$  = horizontal velocity

$$\frac{d\dot{r}}{d\theta} = -\frac{GM}{ur}$$

$\dot{r}$  = radial velocity

$r$  = radial distance from center of gravity

$$\frac{dr}{d\theta} = \frac{\dot{r}r}{u}$$

$GM = 3.986032 \cdot 10^{14}$  (metric units)

Earth mass times gravity constant.

Three-Dimensional Vacuum Trajectories. In this case time has been chosen as independent variable. The equations then are

$$\frac{du}{dt} = -\frac{u\dot{r}}{r}$$

$$\frac{d\chi}{dt} = \frac{u}{r} \sin \chi \tan \lambda$$

$\chi$  = azimuth from north (positive east)

$\lambda$  = geocentric latitude

$$\frac{d\dot{r}}{dt} = \frac{u^2}{r} - \frac{GM}{r^2}$$

$\Lambda$  = longitude

$\omega_e$  = rotational speed of the earth

$$\frac{dr}{dt} = \dot{r}$$



$$\frac{d\lambda}{dt} = \frac{u}{r} \cos \chi$$

$$\frac{d\Lambda}{dt} = \frac{u}{r} \frac{\sin \chi}{\cos \lambda} - \omega_e \quad .$$

**Reentry Equations for State and Adjoint Variables.** These equations were written in a flight path-oriented axis system for a spherical, nonrotating earth and an exponential density-altitude relationship.

$$\dot{v} = -g \sin \gamma - (C_{D_o} + k C_L^n) \frac{S \rho_o}{2m} v^2 e^{-\beta h}$$

$$\dot{\chi} = -\frac{v}{r} \cos \gamma \cos \chi \tan \Lambda + \frac{C_L S \rho_o}{2m} v e^{-\beta h} \frac{\sin \mu}{\cos \gamma}$$

$$\dot{\theta} = \frac{v}{r} \frac{\cos \gamma}{\cos \Lambda} \cos \chi$$

$$\dot{\Lambda} = \frac{v}{r} \cos \gamma \sin \chi$$

$$\dot{h} = v \sin \gamma \quad \quad r = R_E + h$$

The differential equations for the adjoint variables are rather lengthy and are not reproduced here, but are shown in Reference 6. The controls  $C_L$  and  $\mu$  are determined from

$$\sin \mu = \frac{\lambda_\chi}{\omega \cos \gamma} \quad ; \quad \omega = [(\lambda_\chi / \cos \gamma)^2 + \lambda_\gamma^2]^{1/2}$$

$$C_L = [-\omega / (v \lambda_v k n)]^{1/(n-1)} \quad .$$

These are highly nonlinear equations sensitive to small changes in some of the adjoint variables.

Symbols used:

|                      |   |
|----------------------|---|
| $C_{D_0}$            | Zero-lift drag coefficient (set to 0.04)  |
| $C_L$                | Lift coefficient (control)  |
| $g = \frac{GM}{r^2}$ | Local gravitational acceleration  |
| $h$                  | Altitude above sea level  |
| $k$                  | Factor for lift-dependent drag (0.8 and 1.0)  |
| $m$                  | Vehicle mass  |
| $n$                  | Power for lift-dependent drag (1.75)  |
| $R_E$                | Earth radius (6371 km)  |
| $S$                  | Reference area ( $\frac{S}{m} = 100 \text{ [kg/m}^2\text{])}$                               |
| $v$                  | Inertial velocity   |
| $\beta$              | Inverse density scale height (assumed $\beta = 1.45 \cdot 10^{-4} \text{ [m}^{-1}\text{])}$ |
| $\gamma$             | Flight path angle   |
| $\theta$             | Range angle in initial flight direction   |
| $\lambda_i$          | Adjoint variable to $i$   |
| $\mu$                | Bank angle (control)  |
| $\rho_0$             | Density for $h = 0$ (assumed $1.54 \text{ [kg/m}^3\text{])}$                                |
| $\chi$               | Azimuth relative to initial flight direction  |

## Cases Compared

To investigate the stepsize adjustment properties of the integration routines, system 1 has been integrated for eccentricities  $e$  from 0.001 (almost circular) to 0.991 (highly elliptic) with the following initial conditions:

$$u(o) = \left[ \frac{GM}{r} (1+e) \right]^{1/2} \text{ [m/s]}$$

$$\dot{r}(o) = 0$$

$$r(o) = 6\,571\,000 \text{ [m]} \quad .$$

Two complete orbits have been integrated for each set of initial conditions and accuracy requirement.

For the system of differential equations describing three-dimensional motion in vacuum about a rotating earth, an almost circular orbit inclined by 30 deg to the equatorial plane has been chosen:

$$u_o = 7850 \text{ [m/s]}$$

$$\chi_o = 90 \text{ deg}$$

$$\dot{r}_o = 150 \text{ [m/s]}$$

$$r_o = 6\,470\,000 \text{ [m]}$$

$$\lambda_o = 30 \text{ deg}$$

$$\Lambda_o = 0$$

Four complete revolutions have been integrated with the period  $P$  given by

$$P = \frac{2\pi \cdot GM}{\sqrt{2 \cdot E^3}} \quad ; \quad E = (u_o^2 + \dot{r}_o^2)^{1/2} - \frac{GM}{r_o} \quad .$$

With the reentry equations a single three-dimensional skip has been chosen as test case. The initial conditions were

$$\begin{array}{ll}
 v_o = 7950 & \lambda v_o = -0.543197 \\
 \chi_o = 0 & \lambda \chi_o = -2054.06 \\
 \gamma_o = 3 \text{ deg} & \lambda \gamma_o = -46.1287 \\
 \Lambda_o = 0 & \lambda \Lambda_o = 538.79 \\
 h_o = 95\,000 & \lambda h_o = -0.000703044
 \end{array}$$

## Discussion of Results

For weakly elliptic planar orbits ( $e \lesssim 0.25$ ) DIFSYF requires about 30 to 40 percent more evaluations of the differential equations than RK713. The average stepsize is about five times as large for DIFSYF as for RK713 (Figs. 5 and 6).

For accuracies (tolerance per step) required of  $10^{-7}$ , one complete revolution is integrated in two steps by DIFSYF. As the eccentricity increases the optimal stepsizes for integrating the apse-portions and the flanks of the orbit become more and more different. For these cases, the smaller stepsize of RK713 allows a faster stepsize adjustment, resulting in increasing superiority of RK713 (Figs. 7 and 8). For highly eccentric orbits, RK713 requires less than half the evaluations of the right-hand side of the differential equations. The ratio of maximum to minimum stepsize is about equal for both methods ( $\sim 15$  for  $e = 0.991$ ). RK713 seems to be more tolerant against too low accuracy requirements. To obtain approximately periodic orbits, tolerances required for RK713 were  $\sim 10^{-3}$  while DIFSYF required  $10^{-5}$  (Fig. 9).

For the almost circular three-dimensional vacuum orbit DIFSYF takes two integration steps per revolution for  $10^{-5}$  tolerance per step compared to  $10^{-7}$  for planar orbits. Here the sinus-like time traces for  $\chi$  and  $\lambda$  require stepsize adjustment, which causes RK713 to become more superior as the accuracy requirement is increased. The average stepsize for DIFSYF is four times that of RK713. For high accuracies DIFSYF requires twice the number of function evaluations as RK713 (Fig. 10).

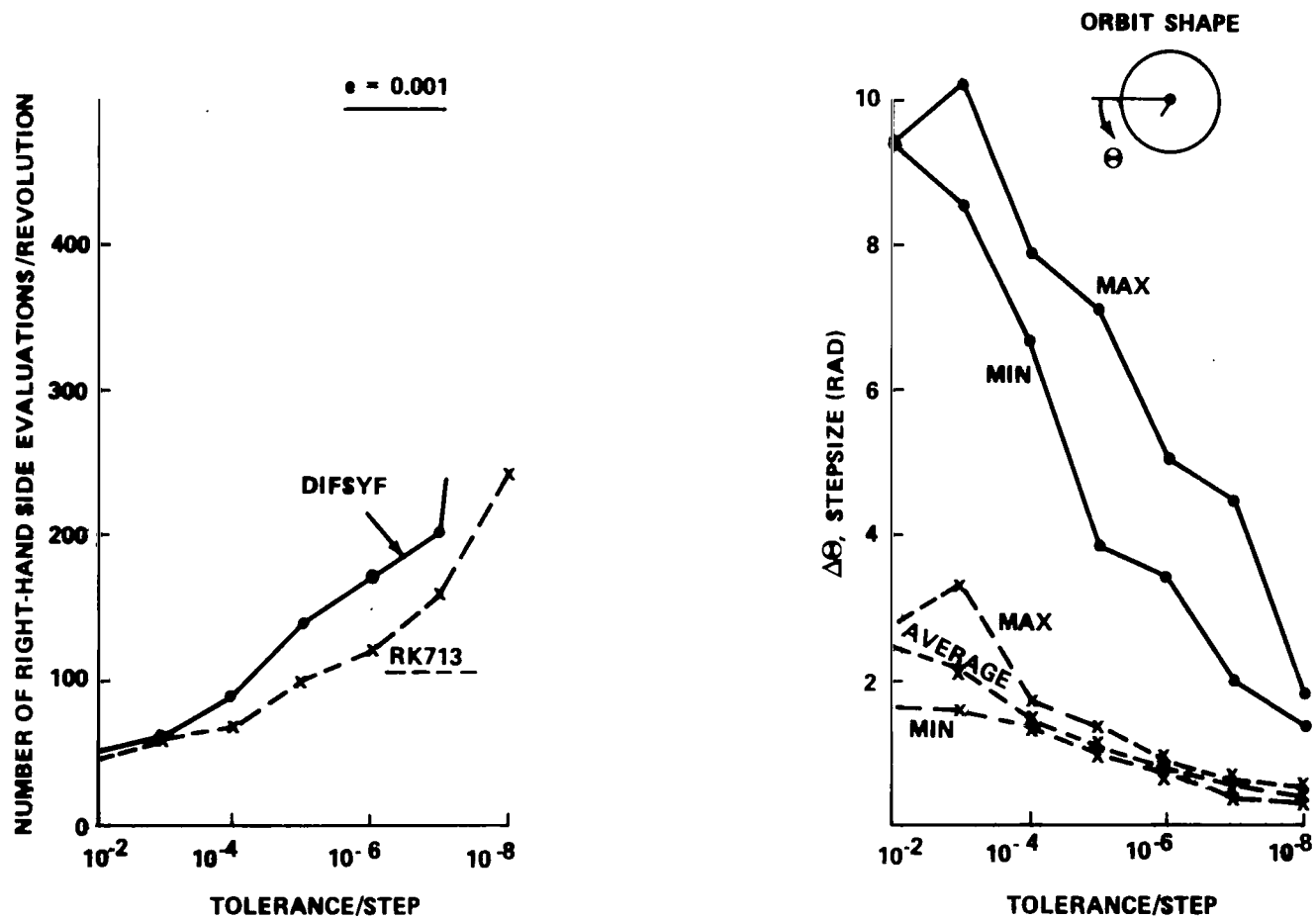


Figure 5. Performance comparison RK713-DIFSYF, planar orbits —  $e=0.001$ , almost circular.

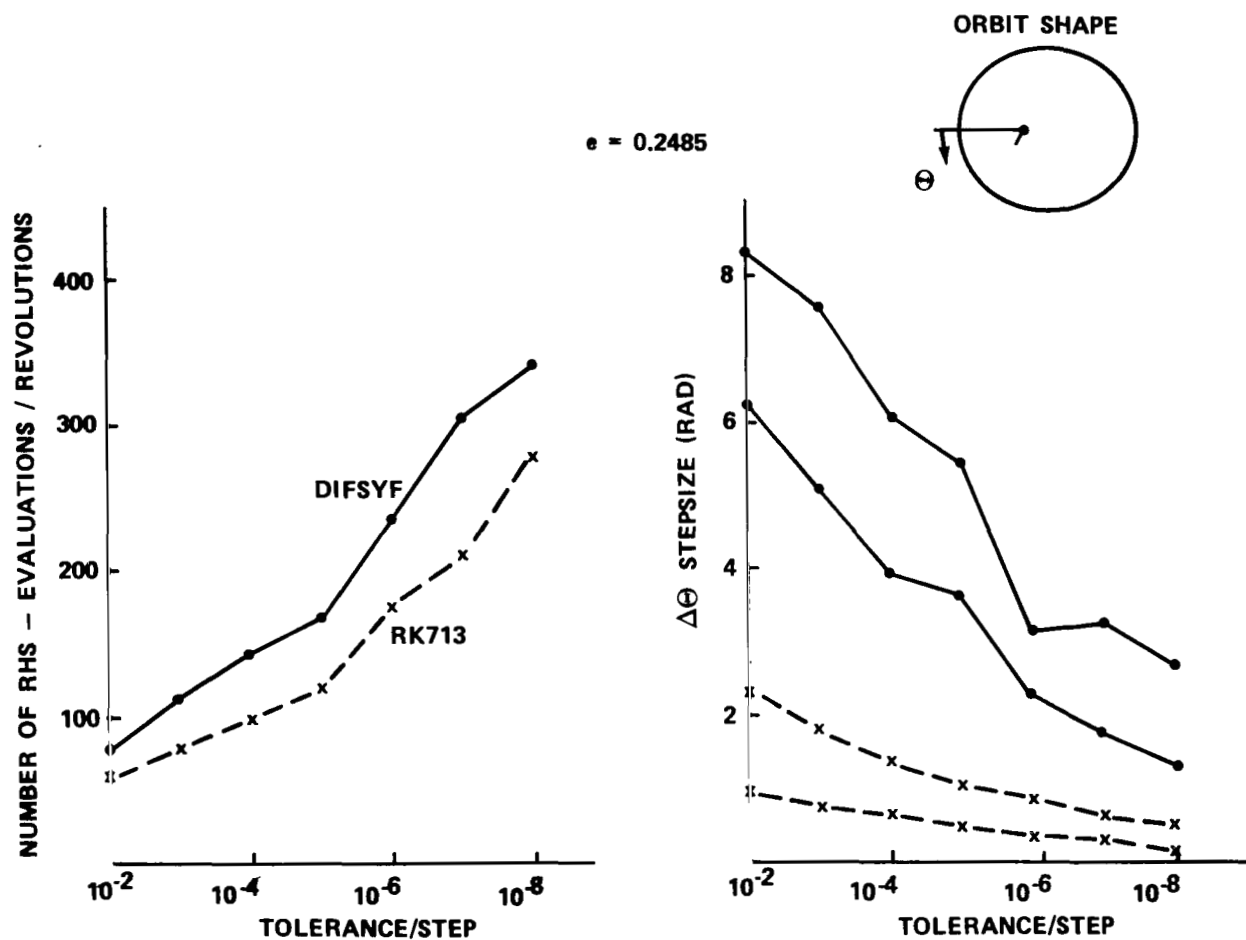


Figure 6. Performance comparison RK713-DIFSYF, planar orbits —  $e = 0.2485$ .

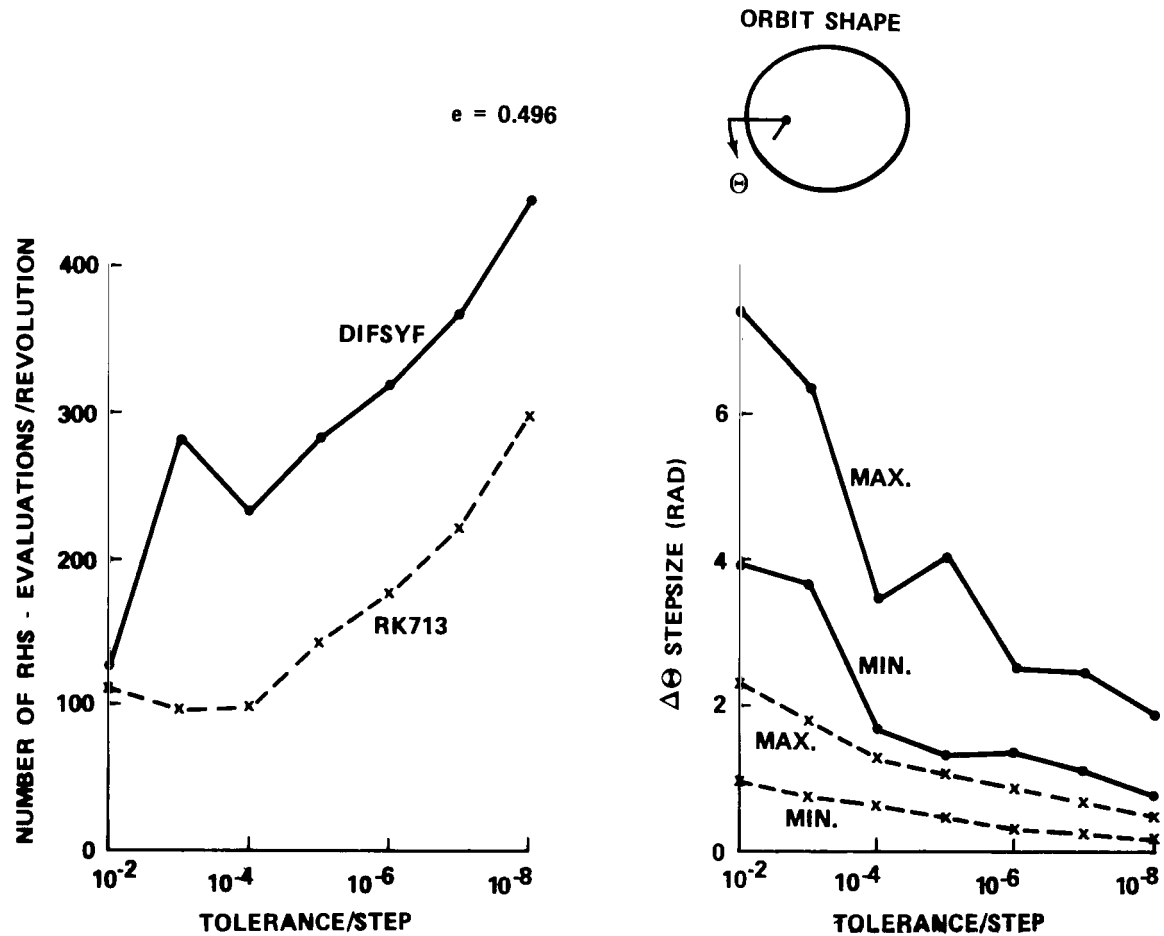


Figure 7. Performance comparison RK713-DIFSYF, planar orbits —  $e = 0.496$ .

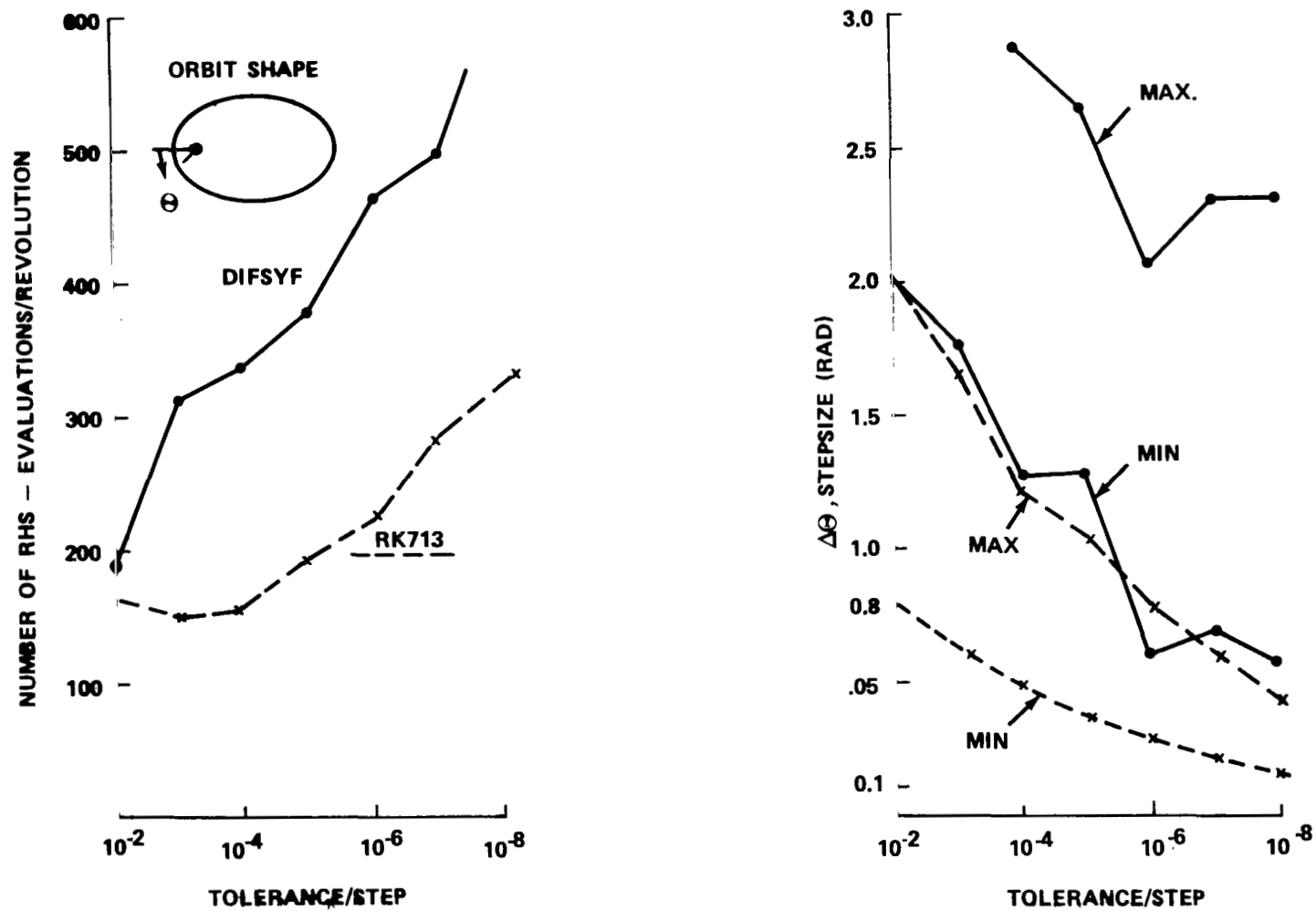


Figure 8. Performance comparison RK713-DIFSYF, planar orbits —  $e=0.7435$ .



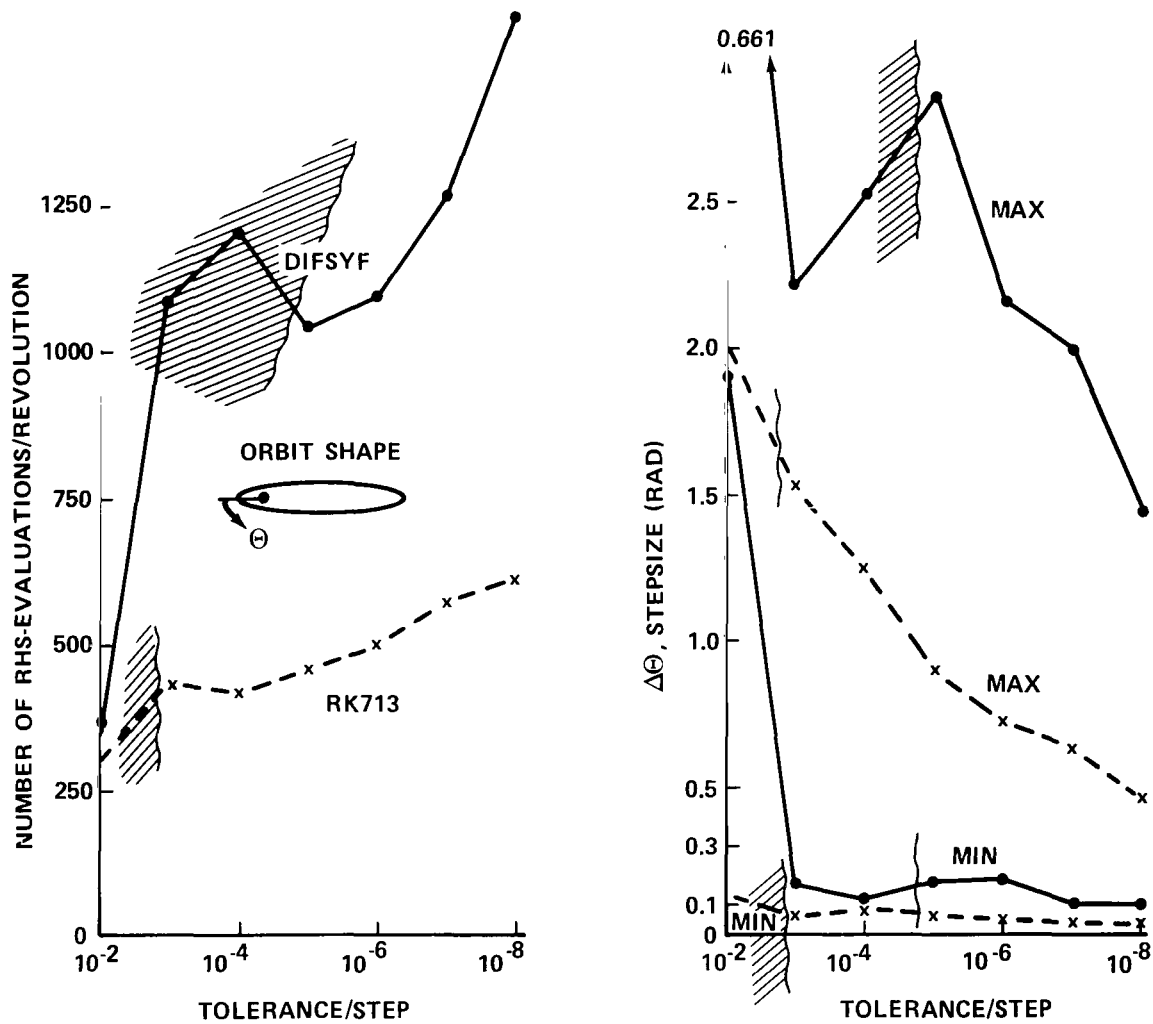


Figure 9. Performance comparison RK713-DIFSYF, planar orbits —  $e=0.991$ ; highly elliptic.

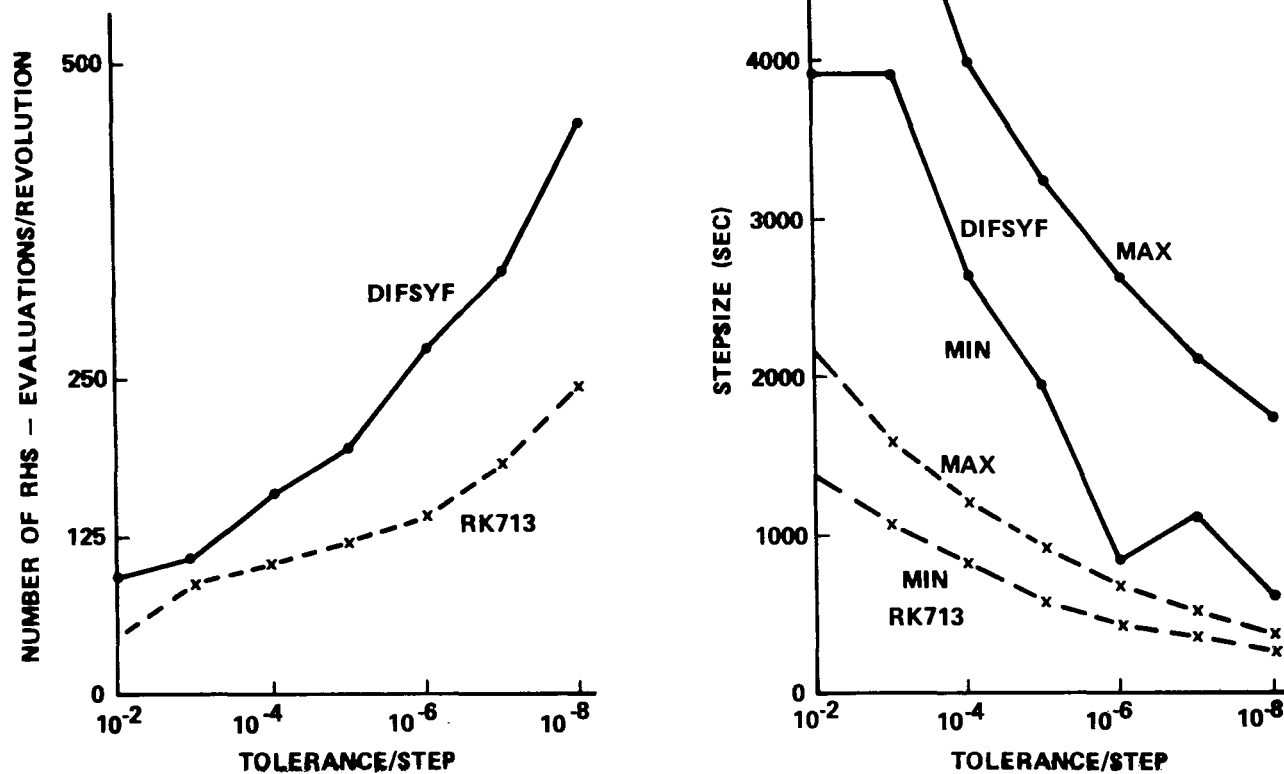


Figure 10. Three-dimensional vacuum orbit (almost circular).

The picture is very similar for the integration of the reentry equations. These test runs were made on a UNIVAC 1108 in double precision for accuracies per step of  $10^{-14}$  to  $10^{-2}$ . For a three-dimensional skip (45-deg plane change) the results are shown in Figure 11. In the usually taken accuracy range  $10^{-5}$  to  $10^{-10}$ , RK713 is slightly superior to DIFSYF. For high accuracy requirements, this superiority becomes more pronounced. It again seems to be a question of stepsize adjustment.

To test the properties of the integration routines with a very demanding trajectory,  $k$  has been changed from 1.0 to 0.8, everything else remaining constant.

The resulting trajectory is a highly oscillating spiral dive with the flight path angle varying from  $-2$  to  $-55$  deg and normal accelerations of 150 g. The results are shown in Figure 12. Again RK713 is superior and takes about one-fourth the stepsize of DIFSYF. It also shows a more benevolent behavior against low accuracy requirements for this numerically unstable system of differential equations. DIFSYF breaks down for accuracies lower than  $10^{-9}$  while RK713 yields good results up to  $10^{-4}$ . The breakdown<sup>2</sup> seems to be related to the minimum stepsize used (lower part of Figure 12). Also, the conservative stepsize control for RK713 discussed in the section of this report entitled Fehlberg's Runge-Kutta Formulas With Stepsize Control contributes to the more stable behavior of RK713 for low accuracy specifications.

Check computations with simple precision (eight digits) for tolerances greater than  $10^{-7}$  were made with RK713. Supposedly because of roundoff errors the number of function evaluations was slightly higher, while the minimum stepsize was slightly smaller. Breakdown occurred at the same tolerance requirement of  $10^{-3}$ .

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2. Breakdown means that when evaluating the right-hand side of the differential equations numbers larger than allowed for in the standard library functions occurred.

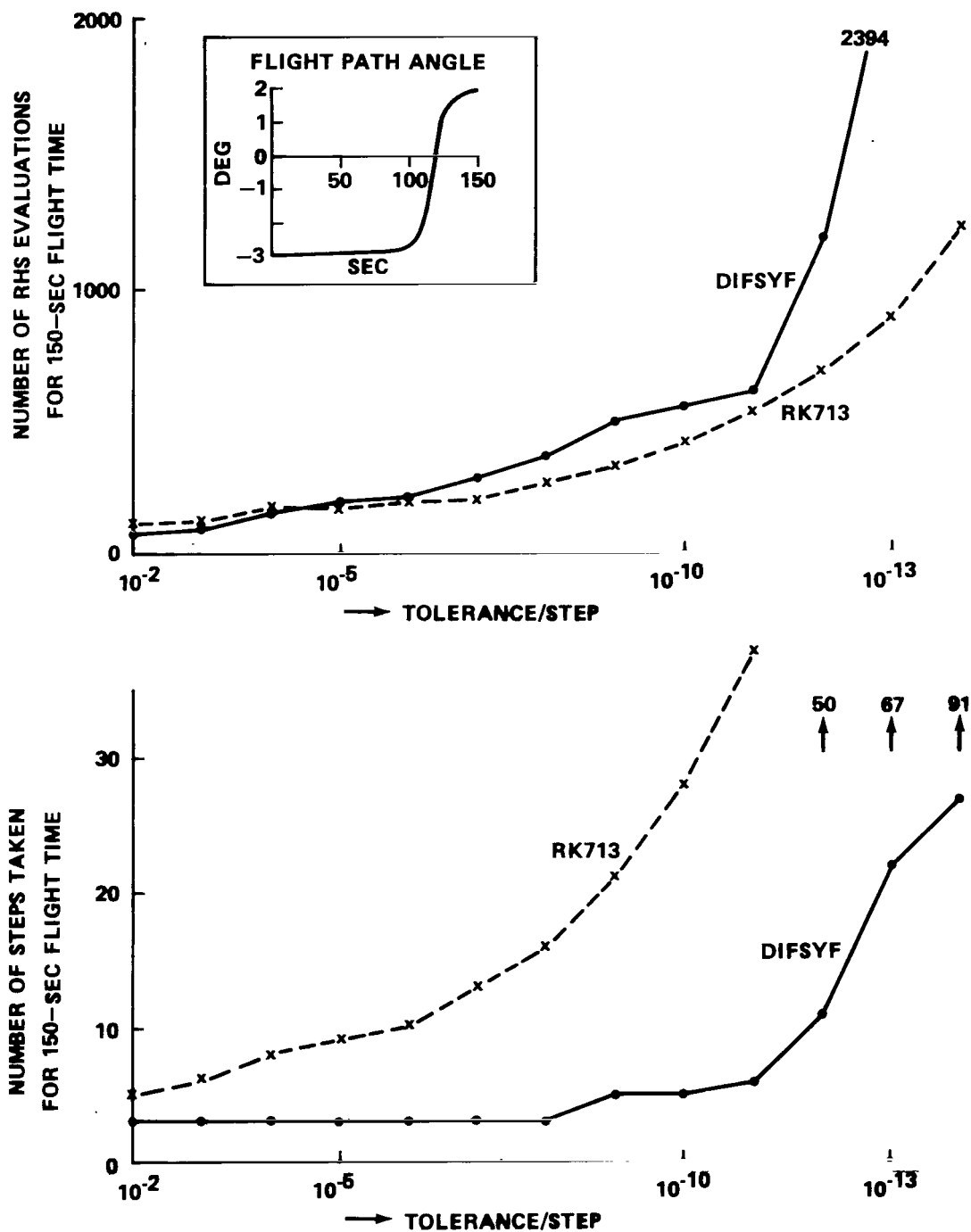


Figure 11. Comparison RK713-DIFSYP, reentry equations — three-dimensional skip trajectory.

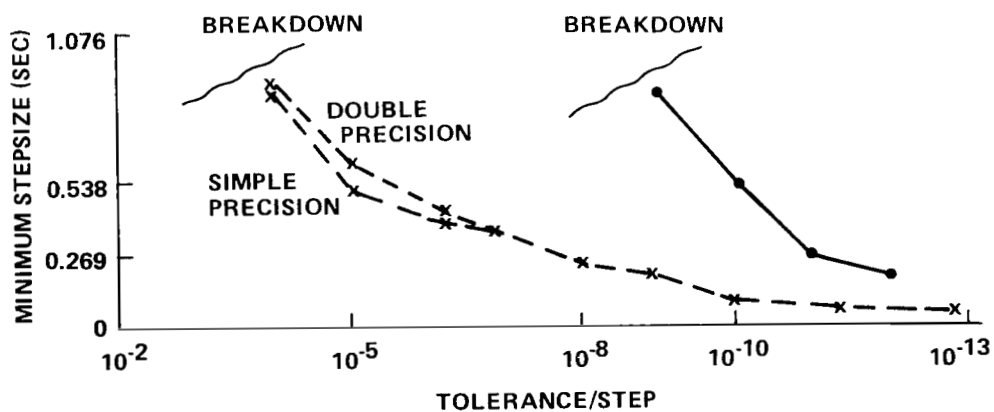
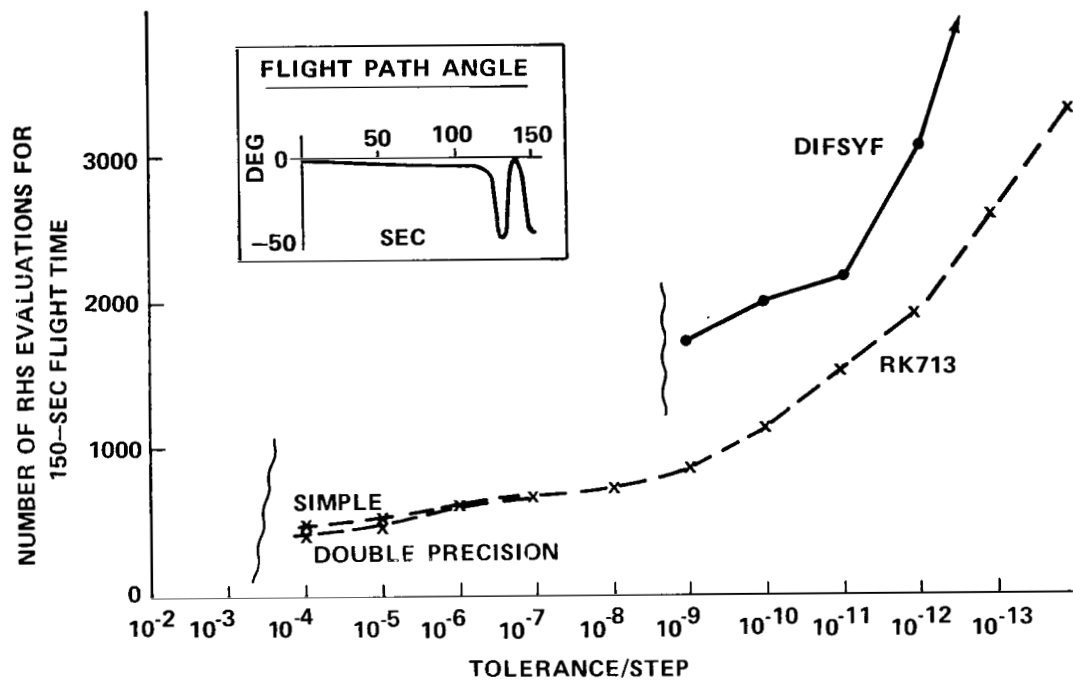


Figure 12. Comparison RK713-DIFSYF, reentry equations — oscillatory spiral dive [highly demanding trajectory ( $K=0.8$ )].

## CONCLUSIONS

The results of this report indicate that Fehlberg's 7-8-13 (RK713) formula with the stepsize control developed in this report is superior to all of the other techniques with which it was compared. That is, it was able to solve all the example problems as fast or faster than any of the other techniques for the entire range of integration accuracies tested. For higher accuracies the superiority of RK713 is particularly evident and in fact RK713 was able to achieve better accuracies on all of the problems than any of the other techniques with which it was compared. Since the rapid solution of systems of differential equations is extremely important to the author of this report, he would appreciate being informed by any of the readers of more efficient techniques for solving these problems.

## APPENDIX A

### SAMPLE COMPUTER PROGRAM LISTINGS AND SAMPLE OUTPUT USING THE FEHLBERG 7-8-13 FORMULA

The following list of symbols is given in order to facilitate the use of the programs:

|      |   |
|------|---|
| DTGI | An estimate of the initial $t$ to be used by the program. After this DTGI is used once, the program is able to compute its own $\Delta t$ so that the value of DTGI used has practically no effect on the number of function evaluations needed for a solution of a particular problem. |
| DTPI | The delta print times for the full print option. That is, if $FPTI < TF$ , then intermediate prints will occur at intervals of DTPI after the print at FPTI.  |
| FPTI | The control for the full-print option of the program. If $FPTI \geq TF$ , then only the initial and final time are printed. If $FPTI < TF$ , then FPTI is the time of the first intermediate print.   |
| KT   | An input control number that allows the program to print every KT integration steps if FPTI and DTPI are both larger than TF. If FPTI and DTPI are less than TF, then KT must be a large number.  |
| TF   | The final time.   |
| TI   | The initial time.   |

TABLE A-1. TEST PROBLEM B1

```

△ASSIGN S=MT0,SI CR,B0 MT1,L0 LP.
△REWIND MT1.
△FORTRAN B0,L0.
  1      DIMENSION X(2)
  2      DIMENSION ALPH(13),BETA(13,12),CH(13)
  3      COMMON ALPH,BETA,CH
  4      20 READ 11,  T1,DTGI,T0L
  5      READ 11,X(1),X(2)
  6      READ 11,TF,FPT1,DTP1
  7      READ 2025,<T
  8      2025 FORMAT(I4)
  9      C      CONSTANTS FOR INTEGRATION SUBROUTINE
 10      DO 60 I=1,13
 11      DO 50 J=1,12
 12      50 BETA(I,J)=0.
 13      ALPH(I)=0.
 14      60 CH(I)=0.
 15      CH(6)=34./105.
 16      CH(7)=9./35.
 17      CH(8)=CH(7)
 18      CH(9)=9./220.
 19      CH(10)=CH(9)
 20      CH(12)=41./840.
 21      CH(13)=CH(12)
 22      ALPH(2)=2./27.
 23      ALPH(3)=1./9.
 24      ALPH(4)=1./6.
 25      ALPH(5)=5./12.
 26      ALPH(6)=.5
 27      ALPH(7)=5./6.
 28      ALPH(8)=1./6.
 29      ALPH(9)=2./3.
 30      ALPH(10)=1./3.
 31      ALPH(11)=1.
 32      ALPH(13)=1.
 33      BETA(2,1)=2./27.
 34      BETA(3,1)=1./36.
 35      BETA(4,1)=1./24.
 36      BETA(5,1)=5./12.
 37      BETA(6,1)=.05
 38      BETA(7,1)=-25./108.
 39      BETA(8,1)=31./300.
 40      BETA(9,1)=2.
 41      BETA(10,1)=-91./108.
 42      BETA(11,1)=2383./4100.
 43      BETA(12,1)=3./205.
 44      BETA(13,1)=-1777./4100.
 45      BETA(3,2)=1./12.
 46      BETA(4,3)=1./8.
 47      BETA(5,3)=-25./16.
 48      BETA(5,4)=-BETA(5,3)
 49      BETA(6,4)=.25
 50      BETA(7,4)=125./108.

```



TABLE A-1. (Continued)

```

51      BETA[9,4]=-53./6.
52      BETA[10,4]=23./108.
53      BETA[11,4]=-341./164.
54      BETA[13,4]=BETA[11,4]
55      BETA[6,5]=.2
56      BETA[7,5]=-65./27.
57      BETA[8,5]=61./225.
58      BETA[9,5]=704./45.
59      BETA[10,5]=-976./135.
60      BETA[11,5]=4496./1025.
61      BETA[13,5]=BETA[11,5]
62      BETA[7,6]=125./54.
63      BETA[8,6]=-2./9.
64      BETA[9,6]=-107./9.
65      BETA[10,6]=311./54.
66      BETA[11,6]=-301./82.
67      BETA[12,6]=-6./41.
68      BETA[13,6]=-289./82.
69      BETA[8,7]=13./900.
70      BETA[9,7]=67./90.
71      BETA[10,7]=-19./60.
72      BETA[11,7]=2133./4100.
73      BETA[12,7]=-3./205.
74      BETA[13,7]=2193./4100.
75      BETA[9,8]=3.
76      BETA[10,8]=17./6.
77      BETA[11,8]=45./82.
78      BETA[12,8]=-3./41.
79      BETA[13,8]=51./82.
80      BETA[10,9]=-1./12.
81      BETA[11,9]=45./164.
82      BETA[12,9]=3./41.
83      BETA[13,9]=33./164.
84      BETA[11,10]=18./41.
85      BETA[12,10]=6./41.
86      BETA[13,10]=12./41.
87      BETA[13,12]=1.
88      NS=0
89      NR=0
90      NST=0
91      NRT=0
92      PRINT 800
93      800  FORMAT (1H1,/,51X,14HINITIAL VALUES)
94      CALL PRINT (TI,X,  FPT1 ,DTP1,DTGI,TOL)
95      T=TI
96      STEP=FPT1
97      DTG=FPT1-T
98      IF (ABS[DTG]-ABS[DTGI])6,6,7
99      7    DTG=DTGI
100     6    NSF=0
101     NSF=0
102     IF (ABS[TF-TI]-ABS[FPT1-TI])112,121,121
103     112  STEP=TF
104     DTG=DTGI

```

TABLE A-1. (Continued)

```

▪ 105 121 CALL INTEGR(T,STEP,DTG,TOL,X, 2,KT ,NSF,NRF)
▪ 106 IF(TF-STEP)161,151,161
▪ 107 161 T=STEP
▪ 108 STEP=T+DTP1
▪ 109 IF(ABS(DTG)-ABS(DTP1))8,8,9
▪ 110 9 DTG=DTP1
▪ 111 8 IF(ABS(TF-T)-ABS(DTP1))132,143,143
▪ 112 132 STEP=TF
▪ 113 IF(ABS(DTG)-ABS(TF+T))143,143,2024
▪ 114 2024 DTG=TF-T
▪ 115 143 PRINT 190,NSF,NRF
▪ 116 190 FORMAT(1H ,///,48X,19HINTERMEDIATE VALUES,3X,I4,1X,19HGOOD STEPS T
▪ 117 1AKEN , ,1X,I4,1X,15HBAD STEPS TAKEN)
▪ 118 CALL PRINT (T ,X, FPT1,DTP1,DTGI,TOL)
▪ 119 NS=NS+NSF
▪ 120 NR=NR+NRF
▪ 121 GO TO 121
▪ 122 151 NS=NS+NSF
▪ 123 NR=NR+NRF
▪ 124 NST=NST+NS
▪ 125 NRT=NRT+NR
▪ 126 PRINT 840,NST,NRT
▪ 127 840 FORMAT(1H0,50X,14HFINAL VALUES ,I4,16HGOOD STEPS TAKEN,I4,
▪ 128 15HBAD STEPS TAKEN)
▪ 129 CALL PRINT(TF,X, FPT1 ,DTP1,DTGI,TOL)
▪ 130 GO TO 20
▪ 131 11 FORMAT(3E18.11)
▪ 132 END

```

## COMMON ALLOCATION

77746 ALPH      77256 BETA      77224 CH

## PROGRAM ALLOCATION

|           |            |            |            |
|-----------|------------|------------|------------|
| 00011 X   | 00015 KT   | 00016 I    | 00017 J    |
| 00020 NS  | 00021 NR   | 00022 NST  | 00023 NRT  |
| 00024 NSF | 00025 NRF  | 00026 TI   | 00030 DTGI |
| 00032 TOL | 00034 TF   | 00036 FPT1 | 00040 DTP1 |
| 00042 T   | 00044 STEP | 00046 DTG  |            |

## SUBPROGRAMS REQUIRED

PRINT      ABS      INTEGR  
THE END

TABLE A-1. (Continued)

```

1      SUBROUTINE INTEGR(TI,T,DTS,TOL ,X,N,KT,NS,NR)
2      DIMENSION      X( 2 ),XE(2 )
3      NT=C
4      NS=C
5      NR=C
6      DTG=DTS
7      TO=TI
8      20  XE(1)=X(1)
9          XE(2)=X(2)
10         STEP=TO+DTG
11         CALL RK713(TO,STEP,DTG,TOL ,X, 2 ,      MS,MR,XE,2 )
12         TO=STEP
13         NS=MS+NS
14         NR=NR+MR
15         DTS=DTG
16         NT=NT+MS+MR
17         IF (STEP-T)240,230,240
18         240 IF (ABS(DTG)-ABS(T- STEP))210,210,260
19         260 DTG=T-STEP
20         210 IF (NT-KT)20,220,220
21         220 T=STEP
22         230 RETURN
23         END

```

PROGRAM ALLOCATION

|          |            |              |          |
|----------|------------|--------------|----------|
| DUMMY X  | 00030 XE   | 00034 INTEGR | 00035 NT |
| DUMMY NS | DUMMY NR   | 00036 MS     | 00037 MR |
| DUMMY KT | 00040 DTG  | DUMMY DTS    | 00042 TO |
| DUMMY TI | 00044 STEP | DUMMY TOL    | DUMMY T  |

SUBPROGRAMS REQUIRED

RK713        ABS  
THE END

TABLE A-1. (Continued)

```

1  SUBROUTINE RK713 (TI,TF,DT,TOL,X,N,NS,NR,XE,M)
2  C SEVENTH ORDER RUNGE-KUTTA INTEGRATION WITH STEPSIZE CONTROL
3  C TF CAN BE GREATER THAN TI OR LESS THAN TI AND RK713 WILL WORK
4  C NS IS THE NUMBER OF SUCCESSFUL STEPS TAKEN
5  C NR IS THE NUMBER OF REJECTED STEPS TAKEN
6  C N IS THE NUMBER OF DIFFERENTIAL EQUATIONS
7  C KT IS MAX NUMBER OF ITERATIONS
8  C ARRAY F STORES THE 13 EVALUATIONS OF THE DIFFERENTIAL EQUATIONS
9  C SUBSCRIPTS FOR ALPHA,BETA, AND CH ARE +1 GREATER THAN FEHLBERGS
10 C F(0) IN FEHLBERGS REPORT IS IN F(1,J)
11 C F(I) IS IN F(I+1,J)
12 C FEHLBERGS REPORT REFERENCED IS NASA TR R-287
13 C PARAMETERS FOR DEQ SUBROUTINE MUST BE STORED IN COMMON
14 C DIMENSIONS MUST AGREE WITH NUMBER OF DIFFERENTIAL EQUATIONS AND
15 C NUMBER OF CONSTANTS IN THE PARTICULAR FEHLBERG FORMULA USED
16 C DIMENSION F(13,2),XDUM(2),TE(2),ALPH(13),
17 C 1BETA(13,12),X(2),CH(13),XE(2)
18 C COMMON ALPH,BETA,CH
19 C T=TI
20 C NS=0
21 C NR=0
22 20 CALL DEQ (X,T,TE)
23 DO 30 I=1,N
24 30 F(1,I)=TE(I)
25 DO 70 K=2,13
26 DO 40 I=1,N
27 40 XDUM(I)=X(I)
28 NN=K-1
29 DO 50 I=1,N
30 DO 50 J=1,NN
31 50 XDUM(I)=XDUM(I)+DT*BETA(K,J)*F(J,I)
32 TDUM=T+ALPH(K)*DT
33 CALL DEQ (XDUM,TDUM,TE)
34 DO 60 I=1,N
35 60 F(K,I)=TE(I)
36 70 CONTINUE
37 ER=0.
38 C M IS AN INPUT VALUE WHICH DETERMINES THE NUMBER OF VARIABLES USED IN
39 C THE ERROR CONTROL LOOP
40 C XE IS AN INPUT VECTOR WITH DIMENSION M WHICH IS USED TO NORMALIZE
41 C THE TRUNCATION ERROR COMPUTATIONS IN THE ERROR CONTROL LOOP
42 DO 120 I=1,M
43 140 TE(I)=DT*(F(1,I)+F(11,I)-F(12,I)-F(13,I))*41./840./XE(I)
44 IF (ABS(TE(I))-ER) 120,120,130
45 130 ER=ABS(TE(I))
46 120 CONTINUE
47 DT1=DT
48 AK=.8
49 IF (ER) 141,142,141
50 142 DT=10.*DT1
51 GO TO 150
52 141 DT=(SQRT(SQRT(TOL/ER)))
53 DT=AK*DT*DT1

```

TABLE A-1. (Continued)

```

*   54      IF (ER .TOL) 150,150,180
*   55      150 TF=T+DT1
*   56      DO 90 I=1,N
*   57      DO 90 L=1,13
*   58      90 X[I]=X[I]+DT1*CH[L]*F[L,I]
*   59      NS=NS+1
*   60      GO TO 230
*   61      180 NR=NR+1
*   62      TF=T
*   63      230 RETURN
*   64      END

```

COMMON ALLOCATION

77746 ALPH      77256 BETA      77224 CH

PROGRAM ALLOCATION

|           |            |             |          |
|-----------|------------|-------------|----------|
| 00037 F   | 00123 XDUM | 00127 TE    | DUMMY X  |
| DUMMY XC  | DUMMY NS   | DUMMY NR    | 00133 I  |
| DUMMY N   | 00134 K    | 00135 NN    | 00136 J  |
| DUMMY M   | 00137 L    | 00140 RK713 | 00142 T  |
| DUMMY TI  | DUMMY DT   | 00144 TDUM  | 00146 ER |
| 00150 DT1 | 00152 AK   | DUMMY TOL   | DUMMY TF |

SUBPROGRAMS REQUIRED

DEQ      ABS      SORT  
THE END

```

*   1      SUBROUTINE PRINT (T,X,    FPT,DTP,DTG,TOL)
*   2      DIMENSION X(2)
*   3      PRINT 1,T,FPT,DTP,DTG,TOL
*   4      1,X[1],X[2]
*   5      1    FORMAT(1H0,5HT    =E18.11,2X,5HFPT =E18.11,2X,5HDTP =E18.11,2X,
*   6      15HDTG =E18.11,2X,5HTOL =E18.11
*   7      2//,6H X[1]=E18.11,2X,5HX[2]=E18.11)
*   8      RETURN
*   9      END

```

PROGRAM ALLOCATION

|           |             |           |           |
|-----------|-------------|-----------|-----------|
| DUMMY X   | 00014 PRINT | DUMMY T   | DUMMY FPT |
| DUMMY DTP | DUMMY DTG   | DUMMY TOL |           |
| THE END   |             |           |           |

TABLE A-1. (Continued)

```

1      SUBROUTINE DEG(X,T,DX)
2      DIMENSION X(2),      DX(2)
3      DX(1)=X(1)**2*X(2)
4      DX(2)=-1./X(1)
5      RETURN
6      END

```

PROGRAM ALLOCATION

```

      DUMMY X          DUMMY DX          00006 DEG
THE END

```

---

```

▲ASSIGN BI=MT1.
▲REWIND MT1.
▲F0RTL0AD BIU.

```

TABLE A-1. (Continued)

NAME ENTRY ORIGIN LAST SIZE/10 COMMON BASE

\*PROGRAM 03507 03477 10636 2656 17063

INITIAL VALUES

T = .0000000000E 00 FPT = .5000000000E 10 DTP = .5000000000E 00 DTG = .1000000000E 00 TOL = .1000000000E 00  
X(1) = .1000000000E 01 X(2) = .1000000000E 01

FINAL VALUES 4000 STEPS TAKEN 0000 STEPS TAKEN

T = .4000000000E 01 FPT = .5000000000E 10 DTP = .5000000000E 00 DTG = .1000000000E 00 TOL = .1000000000E 00  
X(1) = .65333167\*17E 02 X(2) = .21196337833E-01

TABLE A-1. (Continued)

## INITIAL VALUES

T = .0000000000E 00 FPT = .5000000000E 10 DTP = .5000000000E 00 DTG = .1000000000E 00 TOL = .1000000000E-01  
 X(1) = .1000000000E 01 X(2) = .1000000000E 01

## FINAL VALUES 5000 STEPS TAKEN 0BAD STEPS TAKEN

T = .4000000000E 01 FPT = .5000000000E 10 DTP = .5000000000E 00 DTG = .1000000000E 00 TOL = .1000000000E-01  
 X(1) = .55368223122E 02 X(2) = .18555934117E-01

## INITIAL VALUES

T = .0000000000E 00 FPT = .5000000000E 10 DTP = .5000000000E 00 DTG = .1000000000E 00 TOL = .1000000000E-02  
 X(1) = .1000000000E 01 X(2) = .1000000000E 01

## FINAL VALUES 6000 STEPS TAKEN 0BAD STEPS TAKEN

T = .4000000000E 01 FPT = .5000000000E 10 DTP = .5000000000E 00 DTG = .1000000000E 00 TOL = .1000000000E-02  
 X(1) = .54473429571E 02 X(2) = .18272071378E-01



TABLE A-1. (Continued)

INITIAL VALUES  
T = .0000000000E 00 FPT = .5000000000E 10 DTP = .5000000000E 00 DTG = .1000000000E 00 TOL = .1000000000E-03  
X(1) = .1000000000E 01 X(2) = .1000000000E 01

FINAL VALUES 8000 STEPS TAKEN 08AD STEPS TAKEN  
T = .4000000000E 01 FPT = .5000000000E 10 DTP = .5000000000E 00 DTG = .1000000000E 00 TOL = .1000000000E-03  
X(1) = .5454013043E 02 X(2) = .1829596603E-01

---

INITIAL VALUES  
T = .0000000000E 00 FPT = .5000000000E 10 DTP = .5000000000E 00 DTG = .1000000000E 00 TOL = .1000000000E-04  
X(1) = .1000000000E 01 X(2) = .1000000000E 01

FINAL VALUES 9000 STEPS TAKEN 08AD STEPS TAKEN  
T = .4000000000E 01 FPT = .5000000000E 10 DTP = .5000000000E 00 DTG = .1000000000E 00 TOL = .1000000000E-04  
X(1) = .5458532447E 02 X(2) = .1831131843E-01

TABLE A-1. (Continued)

INITIAL VALUES

T = .0000000000E 00 FPT = .5000000000E 10 DTP = .5000000000E 00 DTG = .1000000000E 00 TOL = .1000000000E-05  
X(1) = .1000000000E 01 X(2) = .1000000000E 01

FINAL VALUES 12000 STEPS TAKEN 0BAD STEPS TAKEN

T = .4000000000E 01 FPT = .5000000000E 10 DTP = .5000000000E 00 DTG = .1000000000E 00 TOL = .1000000000E-05  
X(1) = .54595981273E 02 X(2) = .18314910017E-01

---

INITIAL VALUES

T = .0000000000E 00 FPT = .5000000000E 10 DTP = .5000000000E 00 DTG = .1000000000E 00 TOL = .1000000000E-05  
X(1) = .1000000000E 01 X(2) = .1000000000E 01

FINAL VALUES 15000 STEPS TAKEN 0BAD STEPS TAKEN

T = .4000000000E 01 FPT = .5000000000E 10 DTP = .5000000000E 00 DTG = .1000000000E 00 TOL = .1000000000E-05  
X(1) = .54597829417E 02 X(2) = .18315531242E-01

TABLE A-1. (Continued)

|                |   |                 |      |   |                 |     |   |                 |             |      |                 |     |   |                 |
|----------------|---|-----------------|------|---|-----------------|-----|---|-----------------|-------------|------|-----------------|-----|---|-----------------|
| INITIAL VALUES |   |                 |      |   |                 |     |   |                 |             |      |                 |     |   |                 |
| T              | = | .0000000000E 00 | FPT  | = | .5000000000E 10 | DTP | = | .5000000000E 00 | DTG         | =    | .1000000000E 00 | TOL | = | .1000000000E-07 |
| X(1)           | = | .1000000000E 01 | X(2) | = | .1000000000E 01 |     |   |                 |             |      |                 |     |   |                 |
| FINAL VALUES   |   |                 |      |   |                 |     |   | 206880          | STEPS TAKEN | 0840 |                 |     |   |                 |
| T              | = | .4000000000E 01 | FPT  | = | .5000000000E 10 | DTP | = | .5000000000E 00 | DTG         | =    | .1000000000E 00 | TOL | = | .1000000000E-07 |
| X(1)           | = | .5459809949E 02 | X(2) | = | .1831562192E-01 |     |   |                 |             |      |                 |     |   |                 |

---

|                |   |                 |      |   |                 |     |   |                 |             |      |                 |     |   |                 |
|----------------|---|-----------------|------|---|-----------------|-----|---|-----------------|-------------|------|-----------------|-----|---|-----------------|
| INITIAL VALUES |   |                 |      |   |                 |     |   |                 |             |      |                 |     |   |                 |
| T              | = | .0000000000E 00 | FPT  | = | .5000000000E 10 | DTP | = | .5000000000E 00 | DTG         | =    | .1000000000E 00 | TOL | = | .1000000000E-08 |
| X(1)           | = | .1000000000E 01 | X(2) | = | .1000000000E 01 |     |   |                 |             |      |                 |     |   |                 |
| FINAL VALUES   |   |                 |      |   |                 |     |   | 266880          | STEPS TAKEN | 0840 |                 |     |   |                 |
| T              | = | .4000000000E 01 | FPT  | = | .5000000000E 10 | DTP | = | .5000000000E 00 | DTG         | =    | .1000000000E 00 | TOL | = | .1000000000E-08 |
| X(1)           | = | .5459813349E 02 | X(2) | = | .1831563345E-01 |     |   |                 |             |      |                 |     |   |                 |

TABLE A-1. (Continued)

## INITIAL VALUES

T = .0000000000E 00 FPT = .5000000000E 10 DTP = .5000000000E 00 DTG = .1000000000E 00 TOL = .1000000000E-09  
 X(1) = .1000000000E 01 X(2) = .1000000000E 01

## FINAL VALUES 34880 STEPS TAKEN 0BAD STEPS TAKEN

T = .4000000000E 01 FPT = .5000000000E 10 DTP = .5000000000E 00 DTG = .1000000000E 00 TOL = .1000000000E-09  
 X(1) = .5459813532E 02 X(2) = .1831563395E-01

---

## INITIAL VALUES

T = .0000000000E 00 FPT = .5000000000E 10 DTP = .5000000000E 00 DTG = .1000000000E 00 TOL = .1000000000E-10  
 X(1) = .1000000000E 01 X(2) = .1000000000E 01

## FINAL VALUES 44880 STEPS TAKEN 0BAD STEPS TAKEN

T = .4000000000E 01 FPT = .5000000000E 10 DTP = .5000000000E 00 DTG = .1000000000E 00 TOL = .1000000000E-10  
 X(1) = .5459813300E 02 X(2) = .1831563317E-01

TABLE A-1. (Continued)

## INITIAL VALUES

T = .0000000000E 00 FPT = .5000000000E 00 DTP = .5000000000E 00 DTG = .1000000000E 00 TOL = .1000000000E-10  
 X(1) = .1000000000E 01 X(2) = .1000000000E 01

## INTERMEDIATE VALUES 6 GOOD STEPS TAKEN , 0 BAD STEPS TAKEN

T = .5000000000E 00 FPT = .5000000000E 00 DTP = .5000000000E 00 DTG = .1000000000E 00 TOL = .1000000000E-10  
 X(1) = .1648721270E 01 X(2) = .6065306596E 00

## INTERMEDIATE VALUES 7 GOOD STEPS TAKEN , 0 BAD STEPS TAKEN

T = .1000000000E 01 FPT = .5000000000E 00 DTP = .5000000000E 00 DTG = .1000000000E 00 TOL = .1000000000E-10  
 X(1) = .2718281876E 01 X(2) = .3678794409E 00

## INTERMEDIATE VALUES 10 GOOD STEPS TAKEN , 0 BAD STEPS TAKEN

T = .1500000000E 01 FPT = .5000000000E 00 DTP = .5000000000E 00 DTG = .1000000000E 00 TOL = .1000000000E-10  
 X(1) = .4481689060E 01 X(2) = .2231301596E 00

## INTERMEDIATE VALUES 9 GOOD STEPS TAKEN , 0 BAD STEPS TAKEN

T = .2000000000E 01 FPT = .5000000000E 00 DTP = .5000000000E 00 DTG = .1000000000E 00 TOL = .1000000000E-10  
 X(1) = .7389056050E 01 X(2) = .1353352823E 00

## INTERMEDIATE VALUES 8 GOOD STEPS TAKEN , 0 BAD STEPS TAKEN

T = .2500000000E 01 FPT = .5000000000E 00 DTP = .5000000000E 00 DTG = .1000000000E 00 TOL = .1000000000E-10  
 X(1) = .1218249374E 02 X(2) = .8208499712E-01

## INTERMEDIATE VALUES 7 GOOD STEPS TAKEN , 0 BAD STEPS TAKEN

T = .3000000000E 01 FPT = .5000000000E 00 DTP = .5000000000E 00 DTG = .1000000000E 00 TOL = .1000000000E-10  
 X(1) = .2008553592E 02 X(2) = .4978706588E-01

## INTERMEDIATE VALUES 6 GOOD STEPS TAKEN , 0 BAD STEPS TAKEN

T = .3500000000E 01 FPT = .5000000000E 00 DTP = .5000000000E 00 DTG = .1000000000E 00 TOL = .1000000000E-10  
 X(1) = .3311544747E 02 X(2) = .3019737932E-01

TABLE A-1. (Concluded)

|      |                   | FINAL VALUES |                   | 59000 STEPS TAKEN | 0800 STEPS TAKEN  |
|------|-------------------|--------------|-------------------|-------------------|-------------------|
| T    | • .4000000000E 01 | FPT          | • .5000000000E 00 | DTP               | • .5000000000E 00 |
| X(1) | • .5459812991E 02 | X(2)         | • .1831563213E-01 | DTG               | • .1000000000E 00 |
|      |                   |              |                   | TBL               | • .1000000000E-10 |

# TABLE A-2. TEST PROBLEM F1

```

△ASSIGN S=MT0,SI CR,B0 MT1,L0 LP.
△REWIND MT1.
△FORTRAN B0,L0.
*      1      DIMENSION X[4]
*      2      DIMENSION ALPH[13],BETA[13,12],CH[13]
*      3      COMMON GMM
*      4      COMMON ALPH,BETA,CH
*      5      20 READ 11, TI,DTGI
*      6      READ 11,[X[I]],I=1,4]
*      7      READ 11,TF,FPT1,DTP1
*      8      READ 11,GMM,T0L
*      9      READ 2025,KT
*     10      2025 FORMAT[I4]
*     11      C      CONSTANTS FOR INTEGRATION SUBROUTINE
*     12      DO 60 I=1,13
*     13      DO 50 J=1,12
*     14      50 BETA[I,J]=0.
*     15      ALPH[I]=0.
*     16      60 CH[I]=0.
*     17      CH[6]=34./105.
*     18      CH[7]=9./35.
*     19      CH[8]=CH[7]
*     20      CH[9]=9./280.
*     21      CH[10]=CH[9]
*     22      CH[12]=41./840.
*     23      CH[13]=CH[12]
*     24      ALPH[2]=2./27.
*     25      ALPH[3]=1./9.
*     26      ALPH[4]=1./6.
*     27      ALPH[5]=5./12.
*     28      ALPH[6]=.5
*     29      ALPH[7]=5./6.
*     30      ALPH[8]=1./6.
*     31      ALPH[9]=2./3.
*     32      ALPH[10]=1./3.
*     33      ALPH[11]=1.
*     34      ALPH[13]=1.
*     35      BETA[2,1]=2./27.
*     36      BETA[3,1]=1./36.
*     37      BETA[4,1]=1./24.
*     38      BETA[5,1]=5./12.
*     39      BETA[6,1]=.05
*     40      BETA[7,1]=-25./108.
*     41      BETA[8,1]=31./300.
*     42      BETA[9,1]=2.
*     43      BETA[10,1]=-91./108.
*     44      BETA[11,1]=2383./4100.
*     45      BETA[12,1]=3./205.
*     46      BETA[13,1]=-1777./4100.
*     47      BETA[3,2]=1./12.
*     48      BETA[4,3]=1./8.
*     49      BETA[5,3]=-25./16.
*     50      BETA[5,4]=-BETA[5,3]

```

TABLE A-2. (Continued)

|     |  |
|-----|--|
| 51  | BETA[6,4] = .25                          |
| 52  | BETA[7,4] = 125./108.                    |
| 53  | BETA[9,4] = -53./6.                      |
| 54  | BETA[10,4] = 23./108.                    |
| 55  | BETA[11,4] = -341./164.                  |
| 56  | BETA[13,4] = BETA[11,4]                  |
| 57  | BETA[6,5] = .2                           |
| 58  | BETA[7,5] = -65./27.                     |
| 59  | BETA[8,5] = 61./225.                     |
| 60  | BETA[9,5] = 704./45.                     |
| 61  | BETA[10,5] = -976./135.                  |
| 62  | BETA[11,5] = 4496./1025.                 |
| 63  | BETA[13,5] = BETA[11,5]                  |
| 64  | BETA[7,6] = 125./54.                     |
| 65  | BETA[8,6] = -2./9.                       |
| 66  | BETA[9,6] = -107./9.                     |
| 67  | BETA[10,6] = 311./54.                    |
| 68  | BETA[11,6] = -301./82.                   |
| 69  | BETA[12,6] = -6./41.                     |
| 70  | BETA[13,6] = -289./82.                   |
| 71  | BETA[8,7] = 13./900.                     |
| 72  | BETA[9,7] = 67./90.                      |
| 73  | BETA[10,7] = -19./60.                    |
| 74  | BETA[11,7] = 2133./4100.                 |
| 75  | BETA[12,7] = -3./205.                    |
| 76  | BETA[13,7] = 2193./4100.                 |
| 77  | BETA[9,8] = 3.                           |
| 78  | BETA[10,8] = 17./6.                      |
| 79  | BETA[11,8] = 45./82.                     |
| 80  | BETA[12,8] = -3./41.                     |
| 81  | BETA[13,8] = 51./82.                     |
| 82  | BETA[10,9] = -1./12.                     |
| 83  | BETA[11,9] = 45./164.                    |
| 84  | BETA[12,9] = 3./41.                      |
| 85  | BETA[13,9] = 33./164.                    |
| 86  | BETA[11,10] = 18./41.                    |
| 87  | BETA[12,10] = 6./41.                     |
| 88  | BETA[13,10] = 12./41.                    |
| 89  | BETA[13,12] = 1.                         |
| 90  | NS=0                                     |
| 91  | NR=0                                     |
| 92  | NST=0                                    |
| 93  | NRT=0                                    |
| 94  | PRINT 800                                |
| 95  | 800 FORMAT (1H1,/,51X,14HINITIAL VALUES) |
| 96  | CALL PRINT (TI,X, FPT1 ,DTP1,DTGI,TOL)   |
| 97  | T=TI                                     |
| 98  | STEP=FPT1                                |
| 99  | DTG=FPT1-T                               |
| 100 | IF (ABS(DTG)-ABS(DTGI)) 6,6,7            |
| 101 | 7 DTG=DTGI                               |
| 102 | 6 NSF=0                                  |
| 103 | NRF=0                                    |
| 104 | IF (ABS(TF-TI)-ABS(FPT1-TI)) 112,121,121 |



TABLE A-2. (Continued)

```

▪ 105 112 STEP=TF
▪ 106 DTG=DTGI
▪ 107 121 CALL INTEGR[T,STEP,DTG,TOL,X, 4,KT ,NSF,NRF]
▪ 108 IF[TF-STEP]161,151,161
▪ 109 161 T=STEP
▪ 110 STEP=T+DTP1
▪ 111 IF[ABS[DTG]-ABS[DTP1]]8,8,9
▪ 112 9 DTG=DTP1
▪ 113 8 IF[ABS[TF-T]-ABS[DTP1]]132,143,143
▪ 114 132 STFP=TF
▪ 115 IF[ABS[DTG]-ABS[TF-T]]143,143,2024
▪ 116 2024 DTG=TF-T
▪ 117 143 PRINT 190,NSF,NRF
▪ 118 190 FORMAT[1H ,///,48X,19HINTERMEDIATE VALUES,3X,I4,1X,19HG00D STEPS T
▪ 119 1AKFN , ,1X,I4,1X,15HBAD STEPS TAKEN]
▪ 120 CALL PRINT [T ,X, FPT1,DTP1,DTGI,TOL]
▪ 121 NS=NS+NSF
▪ 122 NR=NR+NRF
▪ 123 GO TO 121
▪ 124 151 NS=NS+NSF
▪ 125 NR=NR+NRF
▪ 126 NST=NST+NS
▪ 127 NRT=NRT+NR
▪ 128 PRINT 840,NST,NRT
▪ 129 840 FORMAT[1H0,50X,14HFINAL VALUES ,I4,16HG00D STEPS TAKEN,I4,
▪ 130 115HBAD STEPS TAKEN]
▪ 131 CALL PRINT[TF,X, FPT1 ,DTP1,DTGI,TOL]
▪ 132 GO TO 20
▪ 133 11 FORMAT[3E18.11]
▪ 134 END

```

COMMON ALLOCATION

|           |            |            |          |
|-----------|------------|------------|----------|
| 77776 GMM | 77744 ALPH | 77254 BETA | 77222 CH |
|-----------|------------|------------|----------|

PROGRAM ALLOCATION

|           |            |            |            |
|-----------|------------|------------|------------|
| 00011 X   | 00021 I    | 00022 KT   | 00023 J    |
| 00024 NS  | 00025 NR   | 00026 NST  | 00027 NRT  |
| 00030 NSF | 00031 NRF  | 00032 TI   | 00034 DTGI |
| 00036 TF  | 00040 FPT1 | 00042 DTP1 | 00044 TOL  |
| 00046 T   | 00050 STEP | 00052 DTG  |            |

SUBPROGRAMS REQUIRED

|         |     |        |
|---------|-----|--------|
| PRINT   | ABS | INTEGR |
| THE END |     |        |

TABLE A-2. (Continued)

```

1  SUBROUTINE INTEGR(TI,T,DTS,TOL ,X,N,KT,NS,NR)
2  DIMENSION X(4),XE(4)
3  NT=0
4  NS=0
5  NR=0
6  DTG=DTS
7  TO=TI
8  20  R2=X[1]**2+X[2]**2
9  V2=X[3]**2+X[4]**2
10  R=SQRT(R2)
11  V=SQRT(V2)
12  DO 1 I=1,2
13  XE[I]=R
14  1  XE[I+2]=V
15  STEP=TO+DTG
16  CALL RK713(TO,STEP,DTG,TOL ,X,4 , MS,MR,XE,4 )
17  TO=STEP
18  NS=NS+NS
19  NR=NR+MR
20  DTS=DTG
21  NT=NT+MS+MR
22  IF (STEP-T)240,230,240
23  240 IF (ABS(DTG)-ABS(T- STEP))210,210,260
24  260 DTG=T-STEP
25  210 IF (NT-KT)20,220,220
26  220 T=STEP
27  230 RETURN
28  END

```

PROGRAM ALLOCATION

|          |          |              |           |
|----------|----------|--------------|-----------|
| DUMMY X  | 00031 XE | 00041 INTEGR | 00042 NT  |
| DUMMY NS | DUMMY NR | 00043 I      | 00044 MS  |
| 00045 MR | DUMMY KT | 00046 DTG    | DUMMY DTS |
| 00050 TO | DUMMY TI | 00052 R2     | 00054 V2  |
| 00056 R  | 00060 V  | 00062 STEP   | DUMMY TOL |
| DUMMY T  |          |              |           |

SUBPROGRAMS REQUIRED

SQRT      RK713      ABS  
 THE END

TABLE A-2. (Continued)

```

1  SUBROUTINE RK713 (TI,TF,DT ,TOL ,X,N, NS,NR,XE,M)
2  C SEVENTH ORDER RUNGE-KUTTA INTEGRATION WITH STEPSIZE CONTROL
3  C TF CAN BE GREATER THAN TI OR LESS THAN TI AND RK713 WILL WORK
4  C NS IS THE NUMBER OF SUCCESFULL STEPS TAKEN
5  C NR IS THE NUMBER OF REJECTED STEPS TAKEN
6  C N IS THE NUMBER OF DIFFERENTIAL EQUATIONS
7  C KT IS MAX NUMBER OF ITERATIONS
8  C ARRAY F STORES THE 13 EVALUATIONS OF THE DIFFERENTIAL EQUATIONS
9  C SUBSCRIPTS FOR ALPHA,BETA, AND CH ARE +1 GREATER THAN FEHLBERGS
10 C F[9] IN FEHLBERGS REPORT IS IN F[1,J]
11 C F[1] IS IN F[I+1,J]
12 C FEHLBERGS REPORT REFERENCED IS NASA TR R-287
13 C PARAMETERS FOR DEQ SUBROUTINE MUST BE STORED IN COMMON
14 C DIMENSIONS MUST AGREE WITH NUMBER OF DIFFERENTIAL EQUATIONS AND
15 C NUMBER OF CONSTANTS IN THE PARTICULAR FEHLBERG FORMULA USED
16 C DIMENSION F(13, 4 ),XDUM( 4 ),TE( 4 ) ,ALPH(13),
17 C 1BETA(13,12),X( 4 ),CH(13),XE(4 )
18 C COMMON GMM
19 C COMMON ALPH,BETA,CH
20 C T=TI
21 C NS=0
22 C NR=0
23 C 20 CALL DEQ (X,T,TE)
24 C DO 30 I=1,N
25 C 30 F(1,I)=TE(1)
26 C DO 70 K=2,13
27 C DO 40 I=1,N
28 C 40 XDUM(I)=X(I)
29 C NN=K-1
30 C DO 50 I=1,N
31 C DO 50 J=1,NN
32 C 50 XDUM(I)=XDUM(I)+DT*BETA(K,J)*F(J,I)
33 C TDUM=T+ALPH(K)*DT
34 C CALL DEQ (XDUM,TDUM,TE)
35 C DO 60 I=1,N
36 C 60 F(K,I)=TE(1)
37 C 70 CONTINUE
38 C ER=0.
39 C M IS AN INPUT VALUE WHICH DETERMINES THE NUMBER OF VARIABLES USED IN
40 C THE ERROR CONTROL LOOP
41 C XE IS AN INPUT VECTOR WITH DIMENSION M WHICH IS USED TO NORMALIZE
42 C THE TRUNCATION ERROR COMPUTATIONS IN THE ERROR CONTROL LOOP
43 C DO 120 I=1,M
44 C 140 TE(I)=DT*(F(1,I)+F(11,I)-F(12,I)-F(13,I))*41./840./XE(I)
45 C IF (ABS(TE(I))>ER) 120,120,130
46 C 130 ER=ABS(TE(I))
47 C 120 CONTINUE
48 C DT1=DT
49 C AK=.8
50 C IF(ER)141,142,141
51 C 142 DT=10.*DT1
52 C GO TO 150
53 C 141 DT=[SQRT(SQRT(SQRT(TOL/ER)))]

```

TABLE A-2. (Continued)

```

  54      DT=AK*DT*DT1
  55      IF (ER -TOL) 150,150,180
  56      150 TF=T+DT1
  57      DO 90 I=1,N
  58      DO 90 L=1,13
  59      90 X(I)=X(I)+DT1*CH(L)*F(L,I)
  60      NS=NS+1
  61      GO TO 230
  62      180 NR=NR+1
  63      TF=T
  64      230 RETURN
  65      END

```

## COMMON ALLOCATION

77776 GMM            77744 ALPH            77254 BETA            77222 CH

## PROGRAM ALLOCATION

|           |            |             |          |
|-----------|------------|-------------|----------|
| 00037 F   | 00207 XDUM | 00217 TE    | DUMMY X  |
| DUMMY XE  | DUMMY NS   | DUMMY NR    | 00227 I  |
| DUMMY N   | 00230 K    | 00231 NN    | 00232 J  |
| DUMMY M   | 00233 L    | 00234 RK713 | 00236 T  |
| DUMMY TI  | DUMMY DT   | 00240 TDUM  | 00242 ER |
| 00244 DT1 | 00246 AK   | DUMMY TOL   | DUMMY TF |

## SUBPROGRAMS REQUIRED

DEQ            ABS            SQR

THE END

```

  1      SUBROUTINE PRINT (T,X,  FPT,DTP,DTG,TOL)
  2      DIMENSION X(4)
  3      COMMON GMM
  4      PRINT 1,T,FPT,DTP,DTG,TOL
  5      1,(X(I),I=1,4),GMM
  6      1  FORMAT(1H0,5HT    =E18.11,2X,5HFPT =E18.11,2X,5HDTP =E18.11,2X,
  7      15HDTG =E18.11,2X,5HTOL =E18.11
  8      2,/,6H X(I)=E18.11,2X,5HX(2)=E18.11,2X,5HX(3)=E18.11,2X,5HX(4)=E18.
  9      311,2X,5HGMM =E18.11]
 10      RETURN
 11      END

```

## COMMON ALLOCATION

77776 GMM

## PROGRAM ALLOCATION

|           |           |             |           |
|-----------|-----------|-------------|-----------|
| DUMMY X   | 00014 I   | 00015 PRINT | DUMMY T   |
| DUMMY FPT | DUMMY DTP | DUMMY DTG   | DUMMY TOL |

THE END

TABLE A-2. (Continued)

```

1      SUBROUTINE DEQ(X,T,DX)
2      DIMENSION X(4),      DX(4)
3      COMMON GMM
4      C1=GMM-1.
5      C2=X(1)+GMM
6      C3=C2-1.
7      R12=C2**2+X(2)**2
8      R22=C3**2+X(2)**2
9      R1=SQRT(R12)
10     R2=SQRT(R22)
11     DEN1=C1/R1/R12
12     DEN2=GMM/R2/R22
13     DX(1)=X(3)
14     DX(2)=X(4)
15     DX(3)=X(1)+2.*X(4)+DEN1*C2-DEN2*C3
16     DX(4)=X(2)-2.*X(3)+[DEN1+DEN2]*X(2)
17     RETURN
18     END

```

COMMON ALLOCATION

77776 GMM

PROGRAM ALLOCATION

|          |          |            |            |
|----------|----------|------------|------------|
| DUMMY X  | DUMMY DX | 00013 DEQ  | 00015 C1   |
| 00017 C2 | 00021 C3 | 00023 R12  | 00025 R22  |
| 00027 R1 | 00031 R2 | 00033 DEN1 | 00035 DEN2 |

SUBPROGRAMS REQUIRED

SQRT  
THE END

---

▲ASSIGN BI=MT1.  
 ▲REWIND MT1.  
 ▲F0RTL0AD BIU.

TABLE A-2. (Continued)

NAME      ENTRY ORIGIN    LAST SIZE/10 COMMON    BASE

\*PROGRAM 03507 03477 11237    2913 17061

## INITIAL VALUES

T    .00000000000E 00    FPT    .10000000000E 11    DTP    .10000000000E 01    DTG    .10000000000E 00    TOL    .10000000000E 00  
 X(1)    .99400000000E 00    X(2)    .00000000000E 00    X(3)    .00000000000E 00    X(4)    -.20317326296E 01    GMM    .12277471000E-01

## FINAL VALUES    16000D STEPS TAKEN    12BAD STEPS TAKEN

T    .11124340337E 02    FPT    .10000000000E 11    DTP    .10000000000E 01    DTG    .10000000000E 00    TOL    .10000000000E 00  
 X(1)    .14834414560E 02    X(2)    -.10497957781E 03    X(3)    -.10353360993E 03    X(4)    -.26034096011E 02    GMM    .12277471000E-01

TABLE A-2. (Continued)

INITIAL VALUES

T = .0000000000E 00 FPT = .1000000000E 11 DTP = .1000000000E 01 DTG = .1000000000E 00 TBL = .1000000000E-01  
X(1) = .9940000000E 00 X(2) = .0000000000E 00 X(3) = .0000000000E 00 X(4) = -.2031732629E 01 GMM = .1227747100E-01

FINAL VALUES 210000 STEPS TAKEN 9BAD STEPS TAKEN

T = .11124340337E 02 FPT = .1000000000E 11 DTP = .1000000000E 01 DTG = .1000000000E 00 TBL = .1000000000E-01  
X(1) = .46094020203E 00 X(2) = -.52836197765E 00 X(3) = .63686511117E 00 X(4) = -.3621262282E-01 GMM = .1227747100E-01

INITIAL VALUES

T = .0000000000E 00 FPT = .1000000000E 11 DTP = .1000000000E 01 DTG = .1000000000E 00 TBL = .1000000000E-02  
X(1) = .9940000000E 00 X(2) = .0000000000E 00 X(3) = .0000000000E 00 X(4) = -.2031732629E 01 GMM = .1227747100E-01

FINAL VALUES 380000 STEPS TAKEN 18BAD STEPS TAKEN

T = .11124340337E 02 FPT = .1000000000E 11 DTP = .1000000000E 01 DTG = .1000000000E 00 TBL = .1000000000E-02  
X(1) = .99394287931E 00 X(2) = .20675093999E-03 X(3) = .30556134955E-01 X(4) = -.20396719553E 01 GMM = .1227747100E-01

TABLE A-2. (Continued)

## INITIAL VALUES

T = .0000000000E 00 FPT = .1000000000E 11 DTP = .1000000000E 01 DTG = .1000000000E 00 TOL = .1000000000E-03  
 X(1) = .9940000000E 00 X(2) = .0000000000E 00 X(3) = .0000000000E 00 X(4) = -.20317326296E 01 GMM = .12277471000E-01

## FINAL VALUES 49000 STEPS TAKEN 23BAD STEPS TAKEN

T = .11124340337E 02 FPT = .1000000000E 11 DTP = .1000000000E 01 DTG = .1000000000E 00 TOL = .1000000000E-03  
 X(1) = .99395953273E 00 X(2) = .31208898899E-03 X(3) = .47930058035E-01 X(4) = -.20362226800E 01 GMM = .12277471000E-01

## INITIAL VALUES

T = .0000000000E 00 FPT = .1000000000E 11 DTP = .1000000000E 01 DTG = .1000000000E 00 TOL = .1000000000E-04  
 X(1) = .9940000000E 00 X(2) = .0000000000E 00 X(3) = .0000000000E 00 X(4) = -.20317326296E 01 GMM = .12277471000E-01

## FINAL VALUES 59000 STEPS TAKEN 26BAD STEPS TAKEN

T = .11124340337E 02 FPT = .1000000000E 11 DTP = .1000000000E 01 DTG = .1000000000E 00 TOL = .1000000000E-04  
 X(1) = .99398586886E 00 X(2) = -.52725004310E-04 X(3) = -.85866794634E-02 X(4) = -.20338573144E 01 GMM = .12277471000E-01



TABLE A-2. (Continued)

INITIAL VALUES

T = .0000000000E 00 FPT = .1000000000E 11 DTP = .1000000000E 01 DTG = .1000000000E 00 TOL = .1000000000E-05  
X(1) = .9940000000E 00 X(2) = .0000000000E 00 X(3) = .0000000000E 00 X(4) = -.2031732629E 01 GMM = .1227747100E-01

FINAL VALUES 71600D STEPS TAKEN 298AD STEPS TAKEN

T = .11124340337E 02 FPT = .1000000000E 11 DTP = .1000000000E 01 DTG = .1000000000E 00 TOL = .1000000000E-05  
X(1) = .99399278977E 00 X(2) = -.23143635052E-04 X(3) = -.37964340325E-02 X(4) = -.20328307961E 01 GMM = .1227747100E-01

INITIAL VALUES

T = .0000000000E 00 FPT = .1000000000E 11 DTP = .1000000000E 01 DTG = .1000000000E 00 TOL = .1000000000E-06  
X(1) = .9940000000E 00 X(2) = .0000000000E 00 X(3) = .0000000000E 00 X(4) = -.2031732629E 01 GMM = .1227747100E-01

FINAL VALUES 88600D STEPS TAKEN 288AD STEPS TAKEN

T = .11124340337E 02 FPT = .1000000000E 11 DTP = .1000000000E 01 DTG = .1000000000E 00 TOL = .1000000000E-06  
X(1) = .99399965197E 00 X(2) = -.18169492185E-05 X(3) = -.29115260859E-03 X(4) = -.20317861661E 01 GMM = .1227747100E-01

TABLE A-2. (Continued)

## INITIAL VALUES

T = .0000000000E 00 FPT = .1000000000E 11 DTP = .1000000000E 01 DTG = .1000000000E 00 TBL = .1000000000E-07  
 X(1) = .9940000000E 00 X(2) = .0000000000E 00 X(3) = .0000000000E 00 X(4) = -.2031732629E 01 GMM = .1227747100E-01

## FINAL VALUES 110000 STEPS TAKEN 238AD STEPS TAKEN

T = .11124340337E 02 FPT = .1000000000E 11 DTP = .1000000000E 01 DTG = .1000000000E 00 TBL = .1000000000E-07  
 X(1) = .9939998775E 00 X(2) = -.5286315416E-06 X(3) = -.8548097940E-04 X(4) = -.2031751415E 01 GMM = .1227747100E-01

## INITIAL VALUES

T = .0000000000E 00 FPT = .1000000000E 11 DTP = .1000000000E 01 DTG = .1000000000E 00 TBL = .1000000000E-08  
 X(1) = .9940000000E 00 X(2) = .0000000000E 00 X(3) = .0000000000E 00 X(4) = -.2031732629E 01 GMM = .1227747100E-01

## FINAL VALUES 142000 STEPS TAKEN 178AD STEPS TAKEN

T = .11124340337E 02 FPT = .1000000000E 11 DTP = .1000000000E 01 DTG = .1000000000E 00 TBL = .1000000000E-08  
 X(1) = .9939999299E 00 X(2) = -.25967018380E-06 X(3) = -.42320802805E-04 X(4) = -.2031743354E 01 GMM = .1227747100E-01

TABLE A-2. (Continued)

INITIAL VALUES

T = .0000000000E 00 FPT = .1000000000E 11 DTP = .1000000000E 01 DTG = .1000000000E 00 TBL = .1000000000E-09  
X(1) = .9940000000E 00 X(2) = .0000000000E 00 X(3) = .0000000000E 00 X(4) = -.2031732629E 01 GMM = .1227747100E-01

FINAL VALUES 187600D STEPS TAKEN 11BAD STEPS TAKEN

T = .11124340337E 02 FPT = .1000000000E 11 DTP = .1000000000E 01 DTG = .1000000000E 00 TBL = .1000000000E-09  
X(1) = .99399990570E 00 X(2) = -.33857788284E-06 X(3) = -.55281726418E-04 X(4) = -.20317470697E 01 GMM = .1227747100E-01

---

INITIAL VALUES

T = .0000000000E 00 FPT = .1000000000E 11 DTP = .1000000000E 01 DTG = .1000000000E 00 TBL = .1000000000E-10  
X(1) = .9940000000E 00 X(2) = .0000000000E 00 X(3) = .0000000000E 00 X(4) = -.2031732629E 01 GMM = .1227747100E-01

FINAL VALUES 284600D STEPS TAKEN 6BAD STEPS TAKEN

T = .11124340337E 02 FPT = .1000000000E 11 DTP = .1000000000E 01 DTG = .1000000000E 00 TBL = .1000000000E-10  
X(1) = .99399975458E 00 X(2) = -.89878453735E-06 X(3) = -.14661985150E-03 X(4) = -.20317701969E 01 GMM = .1227747100E-01

TABLE A-2. (Continued)

## INITIAL VALUES

T = .00000000000E 00 FPT = .10000000000E 01 DTP = .10000000000E 01 DTG = .10000000000E 00 TBL = .10000000000E-10  
 X(1) = .99400000000E 00 X(2) = .00000000000E 00 X(3) = .00000000000E 00 X(4) = -.20317326296E 01 GMM = .12277471000E-01

## INTERMEDIATE VALUES

91 GOOD STEPS TAKEN ,

3 BAD STEPS TAKEN

T = .10000000000E 01 FPT = .10000000000E 01 DTP = .10000000000E 01 DTG = .10000000000E 00 TBL = .10000000000E-10  
 X(1) = .51306964334E-01 X(2) = .33261806494E 00 X(3) = -.17725293480E 01 X(4) = .30780989933E 00 GMM = .12277471000E-01

## INTERMEDIATE VALUES

25 GOOD STEPS TAKEN ,

0 BAD STEPS TAKEN

T = .20000000000E 01 FPT = .10000000000E 01 DTP = .10000000000E 01 DTG = .10000000000E 00 TBL = .10000000000E-10  
 X(1) = -.93131060560E 00 X(2) = .24547656687E 00 X(3) = -.43129158899E 00 X(4) = .31006193716E 00 GMM = .12277471000E-01

## INTERMEDIATE VALUES

10 GOOD STEPS TAKEN ,

0 BAD STEPS TAKEN

T = .30000000000E 01 FPT = .10000000000E 01 DTP = .10000000000E 01 DTG = .10000000000E 00 TBL = .10000000000E-10  
 X(1) = -.97614412344E 00 X(2) = .77776335194E 00 X(3) = .33149953230E 00 X(4) = .56540311203E 00 GMM = .12277471000E-01

## INTERMEDIATE VALUES

8 GOOD STEPS TAKEN ,

0 BAD STEPS TAKEN

T = .40000000000E 01 FPT = .10000000000E 01 DTP = .10000000000E 01 DTG = .10000000000E 00 TBL = .10000000000E-10  
 X(1) = -.44270603051E 00 X(2) = .10758527125E 01 X(3) = .57852776536E 00 X(4) = -.49441502508E-01 GMM = .12277471000E-01

## INTERMEDIATE VALUES

21 GOOD STEPS TAKEN ,

0 BAD STEPS TAKEN

T = .50000000000E 01 FPT = .10000000000E 01 DTP = .10000000000E 01 DTG = .10000000000E 00 TBL = .10000000000E-10  
 X(1) = -.18006366263E 00 X(2) = .66950196427E 00 X(3) = -.21786104633E 00 X(4) = -.76073847667E 00 GMM = .12277471000E-01

## INTERMEDIATE VALUES

31 GOOD STEPS TAKEN ,

0 BAD STEPS TAKEN

T = .60000000000E 01 FPT = .10000000000E 01 DTP = .10000000000E 01 DTG = .10000000000E 00 TBL = .10000000000E-10  
 X(1) = -.21621583708E 00 X(2) = -.56708820368E 00 X(3) = .36135607595E 00 X(4) = -.89406724136E 00 GMM = .12277471000E-01

## INTERMEDIATE VALUES

14 GOOD STEPS TAKEN ,

0 BAD STEPS TAKEN

T = .70000000000E 01 FPT = .10000000000E 01 DTP = .10000000000E 01 DTG = .10000000000E 00 TBL = .10000000000E-10  
 X(1) = -.37318694314E 00 X(2) = -.10638047999E 01 X(3) = -.5367403927E 00 X(4) = -.14407343734E 00 GMM = .12277471000E-01

TABLE A-2. (Concluded)

| INTERMEDIATE VALUES |   |                   |      | 9 GOOD STEPS TAKEN , |                   | 0 BAD STEPS TAKEN |   |                   |      |   |                  |     |   |                  |
|---------------------|---|-------------------|------|----------------------|-------------------|-------------------|---|-------------------|------|---|------------------|-----|---|------------------|
| T                   | = | .80000000000E 01  | FPT  | =                    | .10000000000E 01  | DTP               | = | .10000000000E 01  | DTG  | = | .10000000000E 00 | TBL | = | .10000000000E-10 |
| X[1]                | = | -.93001977239E 00 | X[2] | =                    | -.84552159396E 00 | X[3]              | = | -.40885566983E 00 | X[4] | = | .52215234765E 00 | GMM | = | .12277471000E-01 |

| INTERMEDIATE VALUES |   |                   |      | 10 GOOD STEPS TAKEN , |                   | 0 BAD STEPS TAKEN |   |                  |      |   |                  |     |   |                  |
|---------------------|---|-------------------|------|-----------------------|-------------------|-------------------|---|------------------|------|---|------------------|-----|---|------------------|
| T                   | = | .90000000000E 01  | FPT  | =                     | .10000000000E 01  | DTP               | = | .10000000000E 01 | DTG  | = | .10000000000E 00 | TBL | = | .10000000000E-10 |
| X[1]                | = | -.97913676968E 00 | X[2] | =                     | -.29005836245E 00 | X[3]              | = | .33768585314E 00 | X[4] | = | .40414413046E 00 | GMM | = | .12277471000E-01 |

| INTERMEDIATE VALUES |   |                   |      | 19 GOOD STEPS TAKEN , |                   | 0 BAD STEPS TAKEN |   |                  |      |   |                   |     |   |                  |
|---------------------|---|-------------------|------|-----------------------|-------------------|-------------------|---|------------------|------|---|-------------------|-----|---|------------------|
| T                   | = | .10000000000E 02  | FPT  | =                     | .10000000000E 01  | DTP               | = | .10000000000E 01 | DTG  | = | .10000000000E 00  | TBL | = | .10000000000E-10 |
| X[1]                | = | -.16592064157E 00 | X[2] | =                     | -.33677902506E 00 | X[3]              | = | .16497137116E 01 | X[4] | = | -.20392918830E 00 | GMM | = | .12277471000E-01 |

| INTERMEDIATE VALUES |   |                  |      | 31 GOOD STEPS TAKEN , |                  | 1 BAD STEPS TAKEN |   |                  |      |   |                   |     |   |                  |
|---------------------|---|------------------|------|-----------------------|------------------|-------------------|---|------------------|------|---|-------------------|-----|---|------------------|
| T                   | = | .11000000000E 02 | FPT  | =                     | .10000000000E 01 | DTP               | = | .10000000000E 01 | DTG  | = | .10000000000E 00  | TBL | = | .10000000000E-10 |
| X[1]                | = | .89885377499E 00 | X[2] | =                     | .51450563383E-01 | X[3]              | = | .68133864168E 00 | X[4] | = | -.12386168192E 00 | GMM | = | .12277471000E-01 |

| FINAL VALUES |   |                  |      | 303GOOD STEPS TAKEN |                   | 6BAD STEPS TAKEN |   |                   |      |   |                   |     |   |                  |
|--------------|---|------------------|------|---------------------|-------------------|------------------|---|-------------------|------|---|-------------------|-----|---|------------------|
| T            | = | .11124340337E 02 | FPT  | =                   | .10000000000E 01  | DTP              | = | .10000000000E 01  | DTG  | = | .10000000000E 00  | TBL | = | .10000000000E-10 |
| X[1]         | = | .99399975514E 00 | X[2] | =                   | -.89776535297E-06 | X[3]             | = | -.14644710143E-03 | X[4] | = | -.20317701066E 01 | GMM | = | .12277471000E-01 |

TABLE A-3. TEST PROBLEM B12

```

▲ASSIGN S=MT0,SI CR,B0 MT1,L0 LP.
▲REWIND MT1.
▲FORTRAN B0,L0.
  1      DIMENSION X(2)
  2      DIMENSION ALPH(13),BETA(13,12),CH(13)
  3      COMMON ALPH,BETA,CH
  4      20 READ 11, TI,DTGI,T0L
  5      READ 11,X(1),X(2)
  6      READ 11,TF,FPT1,DTP1
  7      READ 2025,KT
  8      2025 FORMAT(I4)
  9      C  CONSTANTS FOR INTEGRATION SUBROUTINE
 10      DO 60 I=1,13
 11      DO 50 J=1,12
 12      50  BETA(I,J)=0.
 13      ALPH(I)=0.
 14      60  CH(I)=0.
 15      CH(6)=34./105.
 16      CH(7)=9./35.
 17      CH(8)=CH(7)
 18      CH(9)=9./280.
 19      CH(10)=CH(9)
 20      CH(12)=41./840.
 21      CH(13)=CH(12)
 22      ALPH(2)=2./27.
 23      ALPH(3)=1./9.
 24      ALPH(4)=1./6.
 25      ALPH(5)=5./12.
 26      ALPH(6)=.5
 27      ALPH(7)=5./6.
 28      ALPH(8)=1./6.
 29      ALPH(9)=2./3.
 30      ALPH(10)=1./3.
 31      ALPH(11)=1.
 32      ALPH(13)=1.
 33      BETA(2,1)=2./27.
 34      BETA(3,1)=1./36.
 35      BETA(4,1)=1./24.
 36      BETA(5,1)=5./12.
 37      BETA(6,1)=.05
 38      BETA(7,1)=-25./108.
 39      BETA(8,1)=31./300.
 40      BETA(9,1)=2.
 41      BETA(10,1)=-91./108.
 42      BETA(11,1)=2383./4100.
 43      BETA(12,1)=3./205.
 44      BETA(13,1)=-1777./4100.
 45      BETA(3,2)=1./12.
 46      BETA(4,3)=1./8.
 47      BETA(5,3)=-25./16.
 48      BETA(5,4)=-BETA(5,3)
 49      BETA(6,4)=.25
 50      BETA(7,4)=125./108.

```

TABLE A-3. (Continued)

```

51      BETA[9,4]=-53./6.
52      BETA[10,4]=23./108.
53      BETA[11,4]=-341./164.
54      BETA[13,4]=BETA[11,4]
55      BETA[6,5]=.2
56      BETA[7,5]=-65./27.
57      BETA[8,5]=61./225.
58      BETA[9,5]=704./45.
59      BETA[10,5]=-976./135.
60      BETA[11,5]=4496./1025.
61      BETA[13,5]=BETA[11,5]
62      BETA[7,6]=125./54.
63      BETA[8,6]=-2./9.
64      BETA[9,6]=-107./9.
65      BETA[10,6]=311./54.
66      BETA[11,6]=-301./82.
67      BETA[12,6]=-6./41.
68      BETA[13,6]=-289./82.
69      BETA[8,7]=13./900.
70      BETA[9,7]=67./90.
71      BETA[10,7]=-19./60.
72      BETA[11,7]=2133./4100.
73      BETA[12,7]=-3./205.
74      BETA[13,7]=2193./4100.
75      BETA[9,8]=3.
76      BETA[10,8]=17./6.
77      BETA[11,8]=45./82.
78      BETA[12,8]=-3./41.
79      BETA[13,8]=51./82.
80      BETA[10,9]=-1./12.
81      BETA[11,9]=45./164.
82      BETA[12,9]=3./41.
83      BETA[13,9]=33./164.
84      BETA[11,10]=18./41.
85      BETA[12,10]=6./41.
86      BETA[13,10]=12./41.
87      BETA[13,12]=1.
88      NS=0
89      NR=0
90      NST=0
91      NRT=0
92      PRINT 800
93      800  FORMAT (1H1,/,51X,14HINITIAL VALUES)
94      CALL PRINT (T1,X,  FPT1 ,DTP1,DTG1,TOL)
95      T=T1
96      STEP=FPT1
97      DTG=FPT1-T
98      IF(ABS(DTG)-ABS(DTG1))6,6,7
99      7    DTG=DTG1
100      6    NSF=0
101      NRF=0
102      IF(ABS(TF-T1)-ABS(FPT1-T1))112,121,121
103      112  STEP=TF
104      DTG=DTG1

```

TABLE A-3. (Continued)

```

▪ 105 121 CALL INTEGR(T,STEP,DTG,TOL,X, 2,KT ,NSF,NRF)
▪ 106 IF[TF-STEP]161,151,161
▪ 107 161 T=STEP
▪ 108 STEP=T+DTP1
▪ 109 IF[ABS(DTG)-ABS(DTP1)]8,8,9
▪ 110 9 DTG=DTP1
▪ 111 8 IF[ABS(TF-T)-ABS(DTP1)]132,143,143
▪ 112 132 STEP=TF
▪ 113 IF[ABS(DTG)-ABS(TF-T)]143,143,2024
▪ 114 2024 DTG=TF-T
▪ 115 143 PRINT 190,NSF,NRF
▪ 116 190 FORMAT(1H ,///,48X,19HINTERMEDIATE VALUES,3X,I4,1X,19H0000 STEPS T
▪ 117 1AKEN , ,1X,I4,1X,15H000 STEPS TAKEN)
▪ 118 CALL PRINT (T ,X, FPT1,DTP1,DTG1,TOL)
▪ 119 NS=NS+NSF
▪ 120 NR=NR+NRF
▪ 121 GO TO 121
▪ 122 151 NS=NS+NSF
▪ 123 NR=NR+NRF
▪ 124 NST=NST+NS
▪ 125 NRT=NRT+NR
▪ 126 PRINT 840,NST,NRT
▪ 127 840 FORMAT(1H0,50X,14HFINAL VALUES ,I4,16H0000 STEPS TAKEN,I4,
▪ 128 115H000 STEPS TAKEN)
▪ 129 CALL PRINT(TF,X, FPT1 ,DTP1,DTG1,TOL)
▪ 130 GO TO 20
▪ 131 11 FORMAT(3E18.11)
▪ 132 END

```

## COMMON ALLOCATION

77746 ALPH      77256 BETA      77224 CH

## PROGRAM ALLOCATION

|           |            |            |            |
|-----------|------------|------------|------------|
| 00011 X   | 00015 KT   | 00016 I    | 00017 J    |
| 00020 NS  | 00021 NR   | 00022 NST  | 00023 NRT  |
| 00024 NSF | 00025 NRF  | 00026 TI   | 00030 DTG1 |
| 00032 TOL | 00034 TF   | 00036 FPT1 | 00040 DTP1 |
| 00042 T   | 00044 STEP | 00046 DTG  |            |

## SUBPROGRAMS REQUIRED

PRINT      ABS      INTEGR  
THE END



TABLE A-3. (Continued)

```

▪      1      SUBROUTINE INTEGR(TI,T,DTS,TOL ,X,N,KT,NS,NR)
▪      2      DIMENSION      X( 2 ),XE(2 )
▪      3      NT=0
▪      4      NS=0
▪      5      NR=0
▪      6      DTG=DTS
▪      7      TO=TI
▪      8      20      XE[1]=X[1]
▪      9      XE[2]=X[2]
▪     10      STEP=TO+DTG
▪     11      CALL RK713(TO,STEP,DTG,TOL ,X, 2 ,      MS,MR,XE,2 )
▪     12      TO=STEP
▪     13      NS=MS+NS
▪     14      NR=NR+MR
▪     15      DTS=DTG
▪     16      NT=NT+MS+MR
▪     17      IF (STEP-T)240,230,240
▪     18      240      IF (ABS(DTG)-ABS(T- STEP))210,210,260
▪     19      260      DTG=T-STEP
▪     20      210      IF (NT-KT)20,220,220
▪     21      220      T=STEP
▪     22      230      RETURN
▪     23      END

```

PROGRAM ALLOCATION

|          |            |              |          |
|----------|------------|--------------|----------|
| DUMMY X  | 00030 XE   | 00034 INTEGR | 00035 NT |
| DUMMY NS | DUMMY NR   | 00036 MS     | 00037 MR |
| DUMMY KT | 00040 DTG  | DUMMY DTS    | 00042 TO |
| DUMMY TI | 00044 STEP | DUMMY TOL    | DUMMY T  |

SUBPROGRAMS REQUIRED

RK713        ABS  
THE END

TABLE A-3. (Continued)

```

1      SUBROUTINE RK713 (TI,TF,DT ,TOL ,X,N, NS,NR,XE,M)
2 C    SEVENTH ORDER RUNGE-KUTTA INTEGRATION WITH STEPSIZE CONTROL
3 C    TF CAN BE GREATER THAN TI OR LESS THAN TI AND RK713 WILL WORK
4 C    NS IS THE NUMBER OF SUCCESSFUL STEPS TAKEN
5 C    NR IS THE NUMBER OF REJECTED STEPS TAKEN
6 C    N IS THE NUMBER OF DIFFERENTIAL EQUATIONS
7 C    KT IS MAX NUMBER OF ITERATIONS
8 C    ARRAY F STORES THE 13 EVALUATIONS OF THE DIFFERENTIAL EQUATIONS
9 C    SUBSCRIPTS FOR ALPHA,BETA, AND CH ARE +1 GREATER THAN FEHLBERGS
10 C   F[I] IN FEHLBERGS REPORT IS IN F[I,J]
11 C   F[I] IS IN F[I+1,J]
12 C   FEHLBERGS REPORT REFERENCED IS NASA TR R-287
13 C   PARAMETERS FOR DEQ SUBROUTINE MUST BE STORED IN COMMON
14 C   DIMENSIONS MUST AGREE WITH NUMBER OF DIFFERENTIAL EQUATIONS AND
15 C   NUMBER OF CONSTANTS IN THE PARTICULAR FEHLBERG FORMULA USED
16     DIMENSION F(13, 2 ),XDUM( 2 ),TE( 2 ) ,ALPH(13),
17     1BETA(13,12),X( 2 ),CH(13),XE(2 )
18     COMMON ALPH,BETA,CH
19     T=TI
20     NS=0
21     NR=0
22     20 CALL DEQ (X,T,TE)
23     DO 30 I=1,N
24     30 F(I,I)=TE(I)
25     DO 70 K=2,13
26     DO 40 J=1,N
27     40 XDUM(I)=X(I)
28     NN=K-1
29     DO 50 I=1,N
30     DO 50 J=1,NN
31     50 XDUM(I)=XDUM(I)+DT*BETA(K,J)*F(J,I)
32     TDUM=T+ALPH(K)*DT
33     CALL DEQ (XDUM,TDUM,TE)
34     DO 60 I=1,N
35     60 F(K,I)=TE(I)
36     70 CONTINUE
37     ER=0.
38 C   M IS AN INPUT VALUE WHICH DETERMINES THE NUMBER OF VARIABLES USED IN
39 C   THE ERROR CONTROL LOOP
40 C   XE IS AN INPUT VECTOR WITH DIMENSION M WHICH IS USED TO NORMALIZE
41 C   THE TRUNCATION ERROR COMPUTATIONS IN THE ERROR CONTROL LOOP
42     DO 120 I=1,M
43     140 TE(I)=DT*(F(1,I)+F(11,I)-F(12,I)-F(13,I))*41./840./XE(I)
44     IF (ABS(TE(I))-ER) 120,120,130
45     130 ER=ABS(TE(I))
46     120 CONTINUE
47     DT1=DT
48     AK=.8
49     IF(ER)141,142,141
50     142 DT=10.*DT1
51     GO TO 150
52     141 DT=(SQRT(SQRT(SQRT(TOL/ER))))
53     DT=AK*DT*DT1

```

TABLE A-3. (Continued)

```

• 54      IF (ER -TOL) 150,150,180
• 55      150 TF=T+DT1
• 56      DO 90 I=1,N
• 57      DO 90 L=1,13
• 58      90 X[I]=X[I]+DT1*CH[L]*F[L,I]
• 59      NS=NS+1
• 60      GO TO 230
• 61      180 NR=NR+1
• 62      TF=T
• 63      230 RETURN
• 64      END

```

COMMON ALLOCATION

77746 ALPH      77256 BETA      77224 CH

PROGRAM ALLOCATION

|           |            |             |          |
|-----------|------------|-------------|----------|
| 00037 F   | 00123 XDUM | 00127 TE    | DUMMY X  |
| DUMMY XE  | DUMMY NS   | DUMMY NR    | 00133 I  |
| DUMMY N   | 00134 K    | 00135 NN    | 00136 J  |
| DUMMY M   | 00137 L    | 00140 RK713 | 00142 T  |
| DUMMY TI  | DUMMY DT   | 00144 TDUM  | 00146 ER |
| 00150 DT1 | 00152 AK   | DUMMY TOL   | DUMMY TF |

SUBPROGRAMS REQUIRED

DEG      ABS      SQRT  
THE END

```

• 1      SUBROUTINE PRINT (T,X,    FPT,DTP,DTG,TOL)
• 2      DIMENSION X(2)
• 3      PRINT 1,T,FPT,DTP,DTG,TOL
• 4      1,X[1],X[2]
• 5      1    FORMAT(1H0,5HT    =E18.11,2X,5HFPT =E18.11,2X,5HDTP =E18.11,2X,
• 6      15HDTG =E18.11,2X,5HTOL =E18.11
• 7      2,/,6H X[1]=E18.11,2X,5HX[2]=E18.11)
• 8      RETURN
• 9      END

```

PROGRAM ALLOCATION

|           |             |           |           |
|-----------|-------------|-----------|-----------|
| DUMMY X   | 00014 PRINT | DUMMY T   | DUMMY FPT |
| DUMMY DTP | DUMMY DTG   | DUMMY TOL |           |
| THE END   |             |           |           |

TABLE A-3. (Continued)

```

*      1      SUBROUTINE DEQ(X,T,DX)
*      2      DIMENSION X(2),      DX(2)
*      3      DX(1)=2.*[1.-X(2)]*X(1)
*      4      DX(2)=-X(2)*[1.-X(1)]
*      5      RETURN
*      6      END

```

PROGRAM ALLOCATION

|         |          |           |
|---------|----------|-----------|
| DUMMY X | DUMMY DX | 00006 DEQ |
| THE END |          |           |

---

```

△ASSIGN BI=MT1.
△REWIND MT1.
△F0RTLOAD BIU.

```

TABLE A-3. (Continued)

NAME ENTRY ORIGIN LAST SIZE/10 COMMON BASE

\*PROGRAM 03507 03477 10024 2262 17063

INITIAL VALUES

T = .0000000000E 00 FPT = .10000000000E 11 DTP = .10000000000E 01 DTG = .10000000000E 00 TBL = .10000000000E 00  
 X(1) = .10000000000E 01 X(2) = .30000000000E 01

FINAL VALUES 41680 STEPS TAKEN 128AD STEPS TAKEN

T = .20000000000E 02 FPT = .10000000000E 11 DTP = .10000000000E 01 DTG = .10000000000E 00 TBL = .10000000000E 00  
 X(1) = .11832482416E 01 X(2) = .22060344093E 00

TABLE A-3. (Continued)

## INITIAL VALUES

T ■ .0000000000E 00 FPT ■ .1000000000E 11 DTP ■ .1000000000E 01 DTG ■ .1000000000E 00 TOL ■ .1000000000E-01  
 X(1) ■ .1000000000E 01 X(2) ■ .3000000000E 01

## FINAL VALUES 32000 STEPS TAKEN 1500 STEPS TAKEN

T ■ .2000000000E 02 FPT ■ .1000000000E 11 DTP ■ .1000000000E 01 DTG ■ .1000000000E 00 TOL ■ .1000000000E-01  
 X(1) ■ .65544148974E 00 X(2) ■ .18534633071E 00

## INITIAL VALUES

T ■ .0000000000E 00 FPT ■ .1000000000E 11 DTP ■ .1000000000E 01 DTG ■ .1000000000E 00 TOL ■ .1000000000E-02  
 X(1) ■ .1000000000E 01 X(2) ■ .3000000000E 01

## FINAL VALUES 32000 STEPS TAKEN 1100 STEPS TAKEN

T ■ .2000000000E 02 FPT ■ .1000000000E 11 DTP ■ .1000000000E 01 DTG ■ .1000000000E 00 TOL ■ .1000000000E-02  
 X(1) ■ .67265193993E 00 X(2) ■ .18597458435E 00

TABLE A-3. (Continued)

INITIAL VALUES

T    ▪ .00000000000E 00   FPT ▪ .10000000000E 11   DTP ▪ .10000000000E 01   DTG ▪ .10000000000E 00   TOL ▪ .10000000000E-03  
X(1)▪ .10000000000E 01   X(2)▪ .30000000000E 01

FINAL VALUES   38000 STEPS TAKEN   1280 STEPS TAKEN

T    ▪ .20000000000E 02   FPT ▪ .10000000000E 11   DTP ▪ .10000000000E 01   DTG ▪ .10000000000E 00   TOL ▪ .10000000000E-03  
X(1)▪ .67546936703E 00   X(2)▪ .18604354748E 00

INITIAL VALUES

T    ▪ .00000000000E 00   FPT ▪ .10000000000E 11   DTP ▪ .10000000000E 01   DTG ▪ .10000000000E 00   TOL ▪ .10000000000E-04  
X(1)▪ .10000000000E 01   X(2)▪ .30000000000E 01

FINAL VALUES   47000 STEPS TAKEN   1180 STEPS TAKEN

T    ▪ .20000000000E 02   FPT ▪ .10000000000E 11   DTP ▪ .10000000000E 01   DTG ▪ .10000000000E 00   TOL ▪ .10000000000E-04  
X(1)▪ .67606171007E 00   X(2)▪ .18607941743E 00

TABLE A-3. (Continued)

INITIAL VALUES

T = .0000000000E 00 FPT = .1000000000E 11 DTP = .1000000000E 01 DTG = .1000000000E 00 TOL = .1000000000E-05  
X(1) = .1000000000E 01 X(2) = .3000000000E 01

FINAL VALUES 61000 STEPS TAKEN 17BAD STEPS TAKEN

T = .2000000000E 02 FPT = .1000000000E 11 DTP = .1000000000E 01 DTG = .1000000000E 00 TOL = .1000000000E-05  
X(1) = .6761812979E 00 X(2) = .18608146993E 00

INITIAL VALUES

T = .0000000000E 00 FPT = .1000000000E 11 DTP = .1000000000E 01 DTG = .1000000000E 00 TOL = .1000000000E-06  
X(1) = .1000000000E 01 X(2) = .3000000000E 01

FINAL VALUES 75000 STEPS TAKEN 13BAD STEPS TAKEN

T = .2000000000E 02 FPT = .1000000000E 11 DTP = .1000000000E 01 DTG = .1000000000E 00 TOL = .1000000000E-06  
X(1) = .67618638263E 00 X(2) = .18608158017E 00



TABLE A-3. (Continued)

INITIAL VALUES

T    • .00000000000E 00    FPT   • .10000000000E 11    DTP   • .10000000000E 01    DTG   • .10000000000E 00    TOL   • .10000000000E-07  
X(1) • .10000000000E 01    X(2) • .30000000000E 01

FINAL VALUES    96000 STEPS TAKEN    11BAD STEPS TAKEN

T    • .20000000000E 02    FPT   • .10000000000E 11    DTP   • .10000000000E 01    DTG   • .10000000000E 00    TOL   • .10000000000E-07  
X(1) • .67618750253E 00    X(2) • .18608160742E 00

---

INITIAL VALUES

T    • .00000000000E 00    FPT   • .10000000000E 11    DTP   • .10000000000E 01    DTG   • .10000000000E 00    TOL   • .10000000000E-08  
X(1) • .10000000000E 01    X(2) • .30000000000E 01

FINAL VALUES    124000 STEPS TAKEN    7BAD STEPS TAKEN

T    • .20000000000E 02    FPT   • .10000000000E 11    DTP   • .10000000000E 01    DTG   • .10000000000E 00    TOL   • .10000000000E-08  
X(1) • .67618758705E 00    X(2) • .18608160911E 00

TABLE A-3. (Continued)

## INITIAL VALUES

T = .0000000000E 00 FPT = .10000000000E 11 DTP = .10000000000E 01 DTG = .10000000000E 00 TOL = .10000000000E-09  
 X(1) = .10000000000E 01 X(2) = .30000000000E 01

## FINAL VALUES 162000 STEPS TAKEN 6840 STEPS TAKEN

T = .20000000000E 02 FPT = .10000000000E 11 DTP = .10000000000E 01 DTG = .10000000000E 00 TOL = .10000000000E-09  
 X(1) = .67618760770E 00 X(2) = .18608160965E 00

## INITIAL VALUES

T = .0000000000E 00 FPT = .10000000000E 11 DTP = .10000000000E 01 DTG = .10000000000E 00 TOL = .10000000000E-10  
 X(1) = .10000000000E 01 X(2) = .30000000000E 01

## FINAL VALUES 217000 STEPS TAKEN 15840 STEPS TAKEN

T = .20000000000E 02 FPT = .10000000000E 11 DTP = .10000000000E 01 DTG = .10000000000E 00 TOL = .10000000000E-10  
 X(1) = .67618761279E 00 X(2) = .18608160961E 00

TABLE A-3. (Continued)

## INITIAL VALUES

T   ▪ .0000000000E 00   FPT  ▪ .1000000000E 01   DTP  ▪ .1000000000E 01   DTG  ▪ .1000000000E 00   TOL  ▪ .1000000000E-10  
 X(1)▪ .1000000000E 01   X(2)▪ .3000000000E 01

## INTERMEDIATE VALUES   14 GOOD STEPS TAKEN ,   2 BAD STEPS TAKEN

T   ▪ .1000000000E 01   FPT  ▪ .1000000000E 01   DTP  ▪ .1000000000E 01   DTG  ▪ .1000000000E 00   TOL  ▪ .1000000000E-10  
 X(1)▪ .7734401615E-01   X(2)▪ .14644481571E 01

## INTERMEDIATE VALUES   10 GOOD STEPS TAKEN ,   0 BAD STEPS TAKEN

T   ▪ .2000000000E 01   FPT  ▪ .1000000000E 01   DTP  ▪ .1000000000E 01   DTG  ▪ .1000000000E 00   TOL  ▪ .1000000000E-10  
 X(1)▪ .84977753179E-01   X(2)▪ .57795270690E 00

## INTERMEDIATE VALUES   9 GOOD STEPS TAKEN ,   0 BAD STEPS TAKEN

T   ▪ .3000000000E 01   FPT  ▪ .1000000000E 01   DTP  ▪ .1000000000E 01   DTG  ▪ .1000000000E 00   TOL  ▪ .1000000000E-10  
 X(1)▪ .29089135173E 00   X(2)▪ .24925317274E 00

## INTERMEDIATE VALUES   9 GOOD STEPS TAKEN ,   1 BAD STEPS TAKEN

T   ▪ .4000000000E 01   FPT  ▪ .1000000000E 01   DTP  ▪ .1000000000E 01   DTG  ▪ .1000000000E 00   TOL  ▪ .1000000000E-10  
 X(1)▪ .14466020925E 01   X(2)▪ .18721896503E 00

## INTERMEDIATE VALUES   14 GOOD STEPS TAKEN ,   0 BAD STEPS TAKEN

T   ▪ .5000000000E 01   FPT  ▪ .1000000000E 01   DTP  ▪ .1000000000E 01   DTG  ▪ .1000000000E 00   TOL  ▪ .1000000000E-10  
 X(1)▪ .40514470448E 01   X(2)▪ .14394903986E 01

## INTERMEDIATE VALUES   19 GOOD STEPS TAKEN ,   0 BAD STEPS TAKEN

T   ▪ .6000000000E 01   FPT  ▪ .1000000000E 01   DTP  ▪ .1000000000E 01   DTG  ▪ .1000000000E 00   TOL  ▪ .1000000000E-10  
 X(1)▪ .17561472735E 00   X(2)▪ .22585894715E 01

## INTERMEDIATE VALUES   12 GOOD STEPS TAKEN ,   1 BAD STEPS TAKEN

T   ▪ .7000000000E 01   FPT  ▪ .1000000000E 01   DTP  ▪ .1000000000E 01   DTG  ▪ .1000000000E 00   TOL  ▪ .1000000000E-10  
 X(1)▪ .65310426683E-01   X(2)▪ .90879526320E 00

TABLE A-3. (Continued)

INTERMEDIATE VALUES      8 GOOD STEPS TAKEN ,      0 BAD STEPS TAKEN  
 T    .8000000000E 01    FPT    .1000000000E 01    DTP    .1000000000E 01    DTG    .1000000000E 00    TOL    .1000000000E-10  
 X(1) .1472268200E 00    X(2) .3667158352E 00

INTERMEDIATE VALUES      8 GOOD STEPS TAKEN ,      0 BAD STEPS TAKEN  
 T    .9000000000E 01    FPT    .1000000000E 01    DTP    .1000000000E 01    DTG    .1000000000E 00    TOL    .1000000000E-10  
 X(1) .6505955588E 00    X(2) .1875738750E 00

INTERMEDIATE VALUES      11 GOOD STEPS TAKEN ,      0 BAD STEPS TAKEN  
 T    .1000000000E 02    FPT    .1000000000E 01    DTP    .1000000000E 01    DTG    .1000000000E 00    TOL    .1000000000E-10  
 X(1) .3144336810E 01    X(2) .3488191625E 00

INTERMEDIATE VALUES      20 GOOD STEPS TAKEN ,      0 BAD STEPS TAKEN  
 T    .1100000000E 02    FPT    .1000000000E 01    DTP    .1000000000E 01    DTG    .1000000000E 00    TOL    .1000000000E-10  
 X(1) .9095183258E 00    X(2) .2996730263E 01

INTERMEDIATE VALUES      14 GOOD STEPS TAKEN ,      0 BAD STEPS TAKEN  
 T    .1200000000E 02    FPT    .1000000000E 01    DTP    .1000000000E 01    DTG    .1000000000E 00    TOL    .1000000000E-10  
 X(1) .7571545536E-01    X(2) .1432714137E 01

INTERMEDIATE VALUES      9 GOOD STEPS TAKEN ,      0 BAD STEPS TAKEN  
 T    .1300000000E 02    FPT    .1000000000E 01    DTP    .1000000000E 01    DTG    .1000000000E 00    TOL    .1000000000E-10  
 X(1) .8672214797E-01    X(2) .5655538011E 00

INTERMEDIATE VALUES      7 GOOD STEPS TAKEN ,      0 BAD STEPS TAKEN  
 T    .1400000000E 02    FPT    .1000000000E 01    DTP    .1000000000E 01    DTG    .1000000000E 00    TOL    .1000000000E-10  
 X(1) .3014693356E 00    X(2) .2451257919E 00

INTERMEDIATE VALUES      9 GOOD STEPS TAKEN ,      0 BAD STEPS TAKEN  
 T    .1500000000E 02    FPT    .1000000000E 01    DTP    .1000000000E 01    DTG    .1000000000E 00    TOL    .1000000000E-10

TABLE A-3. (Concluded)

X(1) = .15034034838E 01 X(2) = .18933981458E 00

INTERMEDIATE VALUES 16 GOOD STEPS TAKEN , 0 BAD STEPS TAKEN

T = .16000000000E 02 FPT = .10000000000E 01 DTP = .10000000000E 01 DTG = .10000000000E 00 TOL = .10000000000E-10  
X(1) = .39579039745E 01 X(2) = .15458979640E 01

INTERMEDIATE VALUES 20 GOOD STEPS TAKEN , 0 BAD STEPS TAKEN

T = .17000000000E 02 FPT = .10000000000E 01 DTP = .10000000000E 01 DTG = .10000000000E 00 TOL = .10000000000E-10  
X(1) = .16560785758E 00 X(2) = .22145804662E 01

INTERMEDIATE VALUES 13 GOOD STEPS TAKEN , 1 BAD STEPS TAKEN

T = .18000000000E 02 FPT = .10000000000E 01 DTP = .10000000000E 01 DTG = .10000000000E 00 TOL = .10000000000E-10  
X(1) = .65624788633E-01 X(2) = .88886887967E 00

INTERMEDIATE VALUES 8 GOOD STEPS TAKEN , 0 BAD STEPS TAKEN

T = .19000000000E 02 FPT = .10000000000E 01 DTP = .10000000000E 01 DTG = .10000000000E 00 TOL = .10000000000E-10  
X(1) = .15174413747E 00 X(2) = .35939068967E 00

FINAL VALUES 239 GOOD STEPS TAKEN 5 BAD STEPS TAKEN

T = .20000000000E 02 FPT = .10000000000E 01 DTP = .10000000000E 01 DTG = .10000000000E 00 TOL = .10000000000E-10  
X(1) = .67618761288E 00 X(2) = .18608160960E 00

TABLE A-4. TEST PROBLEM E22

```

▲ASSIGN S=MT0,SI CR,B0 MT1,L0 LP.
▲REWIND MT1.
▲FORTRAN B0,L0.
  1      DIMENSION X(2)
  2      DIMENSION ALPH(13),BETA(13,12),CH(13)
  3      COMMON ALPH,BETA,CH
  4      20 READ 11,    TI,DTGI,T0L
  5      READ 11,X(1),X(2)
  6      READ 11,TF,FPT1,DTPI
  7      READ 2025,<T
  8      2025  FORMAT(I4)
  9      C      CONSTANTS FOR INTEGRATION SUBROUTINE
 10      DO 60 I=1,13
 11      DO 50 J=1,12
 12      50  BETA(I,J)=0.
 13      ALPH(I)=0.
 14      60  CH(I)=0.
 15      CH(6)=34./105.
 16      CH(7)=9./35.
 17      CH(8)=CH(7)
 18      CH(9)=9./280.
 19      CH(10)=CH(9)
 20      CH(12)=41./840.
 21      CH(13)=CH(12)
 22      ALPH(2)=2./27.
 23      ALPH(3)=1./9.
 24      ALPH(4)=1./6.
 25      ALPH(5)=5./12.
 26      ALPH(6)=.5
 27      ALPH(7)=5./6.
 28      ALPH(8)=1./6.
 29      ALPH(9)=2./3.
 30      ALPH(10)=1./3.
 31      ALPH(11)=1.
 32      ALPH(13)=1.
 33      BETA(2,1)=2./27.
 34      BETA(3,1)=1./36.
 35      BETA(4,1)=1./24.
 36      BETA(5,1)=5./12.
 37      BETA(6,1)=.05
 38      BETA(7,1)=-25./102.
 39      BETA(8,1)=31./200.
 40      BETA(9,1)=2.
 41      BETA(10,1)=-91./108.
 42      BETA(11,1)=2383./100.
 43      BETA(12,1)=3./205.
 44      BETA(13,1)=-1777./4100.
 45      BETA(3,2)=1./12.
 46      BETA(4,3)=1./8.
 47      BETA(5,3)=-25./16.
 48      BETA(5,4)=-BETA(5,3)
 49      BETA(6,4)=.25
 50      BETA(7,4)=125./108.

```

TABLE A-4. (Continued)

```

51      BETA[9,4]=-53./6.
52      BETA[10,4]=23./108.
53      BETA[11,4]=-341./164.
54      BETA[13,4]=BETA[11,4]
55      BETA[6,5]=.2
56      BETA[7,5]=-65./27.
57      BETA[8,5]=61./225.
58      BETA[9,5]=704./45.
59      BETA[10,5]=-976./135.
60      BETA[11,5]=4496./1025.
61      BETA[13,5]=BETA[11,5]
62      BETA[7,6]=125./54.
63      BETA[8,6]=-2./9.
64      BETA[9,6]=-107./9.
65      BETA[10,6]=311./54.
66      BETA[11,6]=-301./82.
67      BETA[12,6]=-6./41.
68      BETA[13,6]=-289./82.
69      BETA[8,7]=13./900.
70      BETA[9,7]=67./90.
71      BETA[10,7]=-19./60.
72      BETA[11,7]=2133./4100.
73      BETA[12,7]=-3./205.
74      BETA[13,7]=2193./4100.
75      BETA[9,8]=3.
76      BETA[10,8]=17./6.
77      BETA[11,8]=45./82.
78      BETA[12,8]=-3./41.
79      BETA[13,8]=51./82.
80      BETA[10,9]=-1./12.
81      BETA[11,9]=45./164.
82      BETA[12,9]=3./41.
83      BETA[13,9]=33./164.
84      BETA[11,10]=18./41.
85      BETA[12,10]=6./41.
86      BETA[13,10]=12./41.
87      BETA[13,12]=1.
88      NS=0
89      NR=0
90      NST=0
91      NRT=0
92      PRINT 800
93      800  FORMAT (1H1,/,51X,14HINITIAL VALUES)
94      CALL PRINT (TI,X,  FPT1 ,DTP1,DTGI,TOL)
95      T=TI
96      STEP=FPT1
97      DTG=FPT1-T
98      IF (ABS(DTG)-ABS(DTGI))6,6,7
99      7    DTG=DTGI
100     6    NSF=0
101     NR=0
102     IF (ABS(TF-TI)-ABS(FPT1-TI))112,121,121
103     112  STEP=TF
104     DTG=DTGI

```

TABLE A-4. (Continued)

```

* 105 121 CALL INTEGR(T,STEP,DTG,TOL,X, 2,KT ,NSF,NRF)
* 106      IF[TF-STEP]161,151,161
* 107 161 T=STEP
* 108      STEP=T+DTP1
* 109      IF[ABS[DTG]-ABS[DTP1]]8,8,9
* 110 9     DTG=DTP1
* 111 8     IF[ABS[TF-T]-ABS[DTP1]]132,143,143
* 112 132   STEP=TF
* 113      IF[ABS[DTG]-ABS[TF-T]]143,143,2024
* 114 2024  DTG=TF-T
* 115 143   PRINT 190,NSF,NRF
* 116      190 FORMAT(1H ,///,48X,19HINTERMEDIATE VALUES,3X,I4,1X,19HG00D STEPS T
* 117      1AKEN , ,1X,I4,1X,15HBAD STEPS TAKEN)
* 118      CALL PRINT [T ,X,  FPT1,DTP1,DTGI,TOL]
* 119      NS=NS+NSF
* 120      NR=NR+NRF
* 121      GO TO 121
* 122 151   NS=NS+NSF
* 123      NR=NR+NRF
* 124      NST=NST+NS
* 125      NRT=NRT+NR
* 126      PRINT 840,NST,NRT
* 127 840   FORMAT(1H0,50X,14HFINAL VALUES ,I4,16HG00D STEPS TAKEN,I4,
* 128      115HBAD STEPS TAKEN)
* 129      CALL PRINT[TF,X,  FPT1 ,DTP1,DTGI,TOL]
* 130      GO TO 20
* 131 11    FORMAT(3E18.11)
* 132      END

```

COMMON ALLOCATION

77746 ALPH      77256 BETA      77224 CH

PROGRAM ALLOCATION

|           |            |            |            |
|-----------|------------|------------|------------|
| 00011 X   | 00015 KT   | 00016 I    | 00017 J    |
| 00020 NS  | 00021 NR   | 00022 NST  | 00023 NRT  |
| 00024 NSF | 00025 NRF  | 00026 TI   | 00030 DTGI |
| 00032 TOL | 00034 TF   | 00036 FPT1 | 00040 DTP1 |
| 00042 T   | 00044 STEP | 00046 DTG  |            |

SUBPROGRAMS REQUIRED

PRINT      ABS      INTEGR  
THE END



TABLE A-4. (Continued)

```

*      1      SUBROUTINE INTEGR(TI,T,DTS,TOL ,X,N,KT,NS,NR)
*      2      DIMENSION      X( 2 ),XE(2 )
*      3      NT=0
*      4      NS=0
*      5      NR=0
*      6      DTG=DTS
*      7      TO=TI
*      8      20      XE[1]=1.
*      9      XE[2]=1.
*     10      STEP=TO+DTG
*     11      CALL RK713(TO,STEP,DTG,TOL ,X, 2 ,      MS,MR,XE,2 )
*     12      TO=STEP
*     13      NS=MS+NS
*     14      NR=NR+MR
*     15      DTS=DTG
*     16      NT=NT+MS+MR
*     17      IF (STEP-T) 240,230,240
*     18      240      IF (ABS(DTG)-ABS(T-STEP)) 210,210,260
*     19      260      DTG=T-STEP
*     20      210      IF (NT-KT) 20,220,220
*     21      220      T=STEP
*     22      230      RETURN
*     23      END

```

PROGRAM ALLOCATION

|          |            |              |          |
|----------|------------|--------------|----------|
| DUMMY X  | 00030 XE   | 00034 INTEGR | 00035 NT |
| DUMMY NS | DUMMY NR   | 00036 MS     | 00037 MR |
| DUMMY KT | 00040 DTG  | DUMMY DTS    | 00042 TO |
| DUMMY TI | 00044 STEP | DUMMY TOL    | DUMMY T  |

SUBPROGRAMS REQUIRED

RK713        ABS  
THE END

TABLE A-4. (Continued)

```

1      SUBROUTINE RK713 (TI,TF,DT ,TOL ,X,N, NS,NR,XE,M)
2      C SEVENTH ORDER RUNGE-KUTTA INTEGRATION WITH STEPSIZE CONTROL
3      C TF CAN BE GREATER THAN TI OR LESS THAN TI AND RK713 WILL WORK
4      C NS IS THE NUMBER OF SUCCESSFUL STEPS TAKEN
5      C NR IS THE NUMBER OF REJECTED STEPS TAKEN
6      C N IS THE NUMBER OF DIFFERENTIAL EQUATIONS
7      C KT IS MAX NUMBER OF ITERATIONS
8      C ARRAY F STORES THE 13 EVALUATIONS OF THE DIFFERENTIAL EQUATIONS
9      C SUBSCRIPTS FOR ALPHA,BETA, AND CH ARE +1 GREATER THAN FEHLBERGS
10     C F[9] IN FEHLBERGS REPORT IS IN F[1,J]
11     C F[1] IS IN F[1+1,J]
12     C FEHLBERGS REPORT REFERENCED IS NASA TR R-287
13     C PARAMETERS FOR DEQ SUBROUTINE MUST BE STORED IN COMMON
14     C DIMENSIONS MUST AGREE WITH NUMBER OF DIFFERENTIAL EQUATIONS AND
15     C NUMBER OF CONSTANTS IN THE PARTICULAR FEHLBERG FORMULA USED
16     C DIMENSION F(13, 2 ),XDUM( 2 ),TE( 2 ) ,ALPH(13),
17     C 1BETA(13,12),X( 2 ),CH(13),XE(2 )
18     C COMMON ALPHA,BETA,CH
19     C T=TI
20     C NS=0
21     C NR=0
22     20 CALL DEQ (X,T,TE)
23     D0 30 I=1,N
24     30 F[1,I]=TE[I]
25     D0 70 K=2,13
26     D0 40 I=1,N
27     40 XDUM[I]=X[I]
28     NN=K-1
29     D0 50 I=1,N
30     D0 50 J=1,NN
31     50 XDUM[I]=XDUM[I]+DT*BETA[K,J]*F[J,I]
32     TDUM=T+ALPH[K]*DT
33     CALL DEQ (XDUM,TDUM,TE)
34     D0 60 I=1,N
35     60 F[K,I]=TE[I]
36     70 CONTINUE
37     ER=0.
38     C M IS AN INPUT VALUE WHICH DETERMINES THE NUMBER OF VARIABLES USED IN
39     C THE ERROR CONTROL LOOP
40     C XE IS AN INPUT VECTOR WITH DIMENSION M WHICH IS USED TO NORMALIZE
41     C THE TRUNCATION ERROR COMPUTATIONS IN THE ERROR CONTROL LOOP
42     D0 120 I=1,M
43     140 TE[I]=DT*(F[1,I]+F[11,I]-F[12,I]-F[13,I])*41./840./XE[I]
44     IF (ABS(TE[I])-ER) 120,120,130
45     130 ER=ABS(TE[I])
46     120 CONTINUE
47     DT1=DT
48     AK=.8
49     IF(ER)141,142,141
50     142 DT=10.*DT1
51     G0 T0 150
52     141 DT=(SQRT(SQRT(TOL/ER)))
53     DT=AK*DT*DT1

```

TABLE A-4. (Continued)

```

• 54      IF (ER -TOL) 150,150,180
• 55      150 TF=T+DT1
• 56      DO 90 I=1,N
• 57      DO 90 L=1,13
• 58      90 X[I]=X[I]+DT1*CH[L]*F[L,I]
• 59      NS=NS+1
• 60      GO TO 230
• 61      180 NR=NR+1
• 62      TF=T
• 63      230 RETURN
• 64      END

```

COMMON ALLOCATION

77746 ALPH      77256 BETA      77224 CH

PROGRAM ALLOCATION

|           |            |             |          |
|-----------|------------|-------------|----------|
| 00037 F   | 00123 XDUM | 00127 TE    | DUMMY X  |
| DUMMY XE  | DUMMY NS   | DUMMY NR    | 00133 I  |
| DUMMY N   | 00134 K    | 00135 NN    | 00136 J  |
| DUMMY M   | 00137 L    | 00140 RK713 | 00142 T  |
| DUMMY TI  | DUMMY DT   | 00144 TDUM  | 00146 ER |
| 00150 DT1 | 00152 AK   | DUMMY TOL   | DUMMY TF |

SUBPROGRAMS REQUIRED

DEG      ABS      SQRT  
THE END

```

• 1      SUBROUTINE PRINT (T,X,    FPT,DTP,DTG,TOL)
• 2      DIMENSION X(2)
• 3      PRINT 1,T,FPT,DTP,DTG,TOL
• 4      1,X[1],X[2]
• 5      1    FORMAT(1H0,5HT    =E18.11,2X,5HFPT =E18.11,2X,5HDTP =E18.11,2X,
• 6      15HDTG =E18.11,2X,5HTOL =E18.11
• 7      2,/,6H X[1]=E18.11,2X,5HX[2]=E18.11)
• 8      RETURN
• 9      END

```

PROGRAM ALLOCATION

|           |             |           |           |
|-----------|-------------|-----------|-----------|
| DUMMY X   | 00014 PRINT | DUMMY T   | DUMMY FPT |
| DUMMY DTP | DUMMY DTG   | DUMMY TOL |           |
| THE END   |             |           |           |

TABLE A-4. (Continued)

|   |   |                              |
|---|---|------------------------------|
| • | 1 | SUBROUTINE DEG(X,T,DX)       |
| • | 2 | DIMENSION X(2), DX(2)        |
| • | 3 | DX(1)=X(2)                   |
| • | 4 | DX(2)=[1,-X(1)**2]*X(2)-X(1) |
| • | 5 | RETURN                       |
| • | 6 | END                          |

PROGRAM ALLOCATION

|         |          |           |
|---------|----------|-----------|
| DUMMY X | DUMMY DX | 00006 DEG |
| THE END |          |           |

---

▲ASSIGN BI=MT1.  
 ▲REWIND MT1.  
 ▲F0RTL0AD BIU.

TABLE A-4. (Continued)

NAME ENTRY ORIGIN LAST SIZE/10 COMMON BASE

\*PROGRAM 03507 03477 10637 2657 17063

INITIAL VALUES

T . .00000000000E 00 FPT . .10000000000E 11 DTP . .10000000000E 01 DTG . .10000000000E 00 TOL . .10000000000E 00  
X(1) . .20000000000E 01 X(2) . .00000000000E 00

FINAL VALUES 35689D STEPS TAKEN 138AD STEPS TAKEN

T . .20000000000E 02 FPT . .10000000000E 11 DTP . .10000000000E 01 DTG . .10000000000E 00 TOL . .10000000000E 00  
X(1) . .19735582007E 01 X(2) . .41827137254E 00

TABLE A-4. (Continued)

## INITIAL VALUES

T = .0000000000E 00 FPT = .1000000000E 11 DTP = .1000000000E 01 DTG = .1000000000E 00 TBL = .1000000000E-01  
 X(1) = .2000000000E 01 X(2) = .0000000000E 00

## FINAL VALUES 36000 STEPS TAKEN 15BAD STEPS TAKEN

T = .2000000000E 02 FPT = .1000000000E 11 DTP = .1000000000E 01 DTG = .1000000000E 00 TBL = .1000000000E-01  
 X(1) = .2006964366E 01 X(2) = -.7184504901E-01

## INITIAL VALUES

T = .0000000000E 00 FPT = .1000000000E 11 DTP = .1000000000E 01 DTG = .1000000000E 00 TBL = .1000000000E-02  
 X(1) = .2000000000E 01 X(2) = .0000000000E 00

## FINAL VALUES 37000 STEPS TAKEN 17BAD STEPS TAKEN

T = .2000000000E 02 FPT = .1000000000E 11 DTP = .1000000000E 01 DTG = .1000000000E 00 TBL = .1000000000E-02  
 X(1) = .2008187402E 01 X(2) = -.4173766412E-01

TABLE A-4. (Continued)

INITIAL VALUES

T = .0000000000E 00 FPT = .1000000000E 11 DTP = .1000000000E 01 DTG = .1000000000E 00 TOL = .1000000000E-03  
X(1) = .2000000000E 01 X(2) = .0000000000E 00

FINAL VALUES 46000 STEPS TAKEN 1600 STEPS TAKEN

T = .2000000000E 02 FPT = .1000000000E 11 DTP = .1000000000E 01 DTG = .1000000000E 00 TOL = .1000000000E-03  
X(1) = .2008164429E 01 X(2) = -.42197190845E-01

INITIAL VALUES

T = .0000000000E 00 FPT = .1000000000E 11 DTP = .1000000000E 01 DTG = .1000000000E 00 TOL = .1000000000E-04  
X(1) = .2000000000E 01 X(2) = .0000000000E 00

FINAL VALUES 56000 STEPS TAKEN 1800 STEPS TAKEN

T = .2000000000E 02 FPT = .1000000000E 11 DTP = .1000000000E 01 DTG = .1000000000E 00 TOL = .1000000000E-04  
X(1) = .20081537041E 01 X(2) = -.42504030549E-01

TABLE A-4. (Continued)

## INITIAL VALUES

T = .0000000000E 00 FPT = .1000000000E 11 DTP = .1000000000E 01 DTG = .1000000000E 00 TOL = .1000000000E-05  
 X(1) = .2000000000E 01 X(2) = .0000000000E 00

## FINAL VALUES 70600D STEPS TAKEN 188AD STEPS TAKEN

T = .2000000000E 02 FPT = .1000000000E 11 DTP = .1000000000E 01 DTG = .1000000000E 00 TOL = .1000000000E-05  
 X(1) = .20081500503E 01 X(2) = -.42508225085E-01

---

## INITIAL VALUES

T = .0000000000E 00 FPT = .1000000000E 11 DTP = .1000000000E 01 DTG = .1000000000E 00 TOL = .1000000000E-06  
 X(1) = .2000000000E 01 X(2) = .0000000000E 00

## FINAL VALUES 89600D STEPS TAKEN 218AD STEPS TAKEN

T = .2000000000E 02 FPT = .1000000000E 11 DTP = .1000000000E 01 DTG = .1000000000E 00 TOL = .1000000000E-06  
 X(1) = .20081497884E 01 X(2) = -.42508639103E-01



TABLE A-4. (Continued)

INITIAL VALUES

T = .0000000000E 00 FPT = .1000000000E 11 DTP = .1000000000E 01 DTG = .1000000000E 00 TOL = .1000000000E-07  
X(1) = .2000000000E 01 X(2) = .0000000000E 00

FINAL VALUES 112000 STEPS TAKEN 188AD STEPS TAKEN

T = .2000000000E 02 FPT = .1000000000E 11 DTP = .1000000000E 01 DTG = .1000000000E 00 TOL = .1000000000E-07  
X(1) = .20081497650E 01 X(2) = -.4250887041E-01

---

INITIAL VALUES

T = .0000000000E 00 FPT = .1000000000E 11 DTP = .1000000000E 01 DTG = .1000000000E 00 TOL = .1000000000E-08  
X(1) = .2000000000E 01 X(2) = .0000000000E 00

FINAL VALUES 144000 STEPS TAKEN 168AD STEPS TAKEN

T = .2000000000E 02 FPT = .1000000000E 11 DTP = .1000000000E 01 DTG = .1000000000E 00 TOL = .1000000000E-08  
X(1) = .20081497620E 01 X(2) = -.42508880677E-01

TABLE A-4. (Continued)

INITIAL VALUES

T = .0000000000E 00 FPT = .1000000000E 11 DTP = .1000000000E 01 DTG = .1000000000E 00 TBL = .1000000000E-09  
X(1) = .2000000000E 01 X(2) = .0000000000E 00

FINAL VALUES 186000 STEPS TAKEN 12BAD STEPS TAKEN

T = .2000000000E 02 FPT = .1000000000E 11 DTP = .1000000000E 01 DTG = .1000000000E 00 TBL = .1000000000E-09  
X(1) = .20081497617E 01 X(2) = -.42508884626E-01

---

INITIAL VALUES

T = .0000000000E 00 FPT = .1000000000E 11 DTP = .1000000000E 01 DTG = .1000000000E 00 TBL = .1000000000E-10  
X(1) = .2000000000E 01 X(2) = .0000000000E 00

FINAL VALUES 242000 STEPS TAKEN 4BAD STEPS TAKEN

T = .2000000000E 02 FPT = .1000000000E 11 DTP = .1000000000E 01 DTG = .1000000000E 00 TBL = .1000000000E-10  
X(1) = .20081497615E 01 X(2) = -.42508886517E-01

TABLE A-4. (Continued)

INITIAL VALUES

T = .0000000000E 00 FPT = .1000000000E 01 DTP = .1000000000E 01 DTG = .1000000000E 00 TOL = .1000000000E-10  
X(1) = .2000000000E 01 X(2) = .0000000000E 00

INTERMEDIATE VALUES 13 GOOD STEPS TAKEN , 1 BAD STEPS TAKEN

T = .1000000000E 01 FPT = .1000000000E 01 DTP = .1000000000E 01 DTG = .1000000000E 00 TOL = .1000000000E-10  
X(1) = .15081442366E 01 X(2) = -.78021807484E 00

INTERMEDIATE VALUES 13 GOOD STEPS TAKEN , 0 BAD STEPS TAKEN

T = .2000000000E 01 FPT = .1000000000E 01 DTP = .1000000000E 01 DTG = .1000000000E 00 TOL = .1000000000E-10  
X(1) = .32331666552E 00 X(2) = -.18329745696E 01

INTERMEDIATE VALUES 16 GOOD STEPS TAKEN , 1 BAD STEPS TAKEN

T = .3000000000E 01 FPT = .1000000000E 01 DTP = .1000000000E 01 DTG = .1000000000E 00 TOL = .1000000000E-10  
X(1) = -.18660739124E 01 X(2) = -.10210603358E 01

INTERMEDIATE VALUES 14 GOOD STEPS TAKEN , 0 BAD STEPS TAKEN

T = .4000000000E 01 FPT = .1000000000E 01 DTP = .1000000000E 01 DTG = .1000000000E 00 TOL = .1000000000E-10  
X(1) = -.17417683244E 01 X(2) = .62466616401E 00

INTERMEDIATE VALUES 9 GOOD STEPS TAKEN , 0 BAD STEPS TAKEN

T = .5000000000E 01 FPT = .1000000000E 01 DTP = .1000000000E 01 DTG = .1000000000E 00 TOL = .1000000000E-10  
X(1) = -.83707745035E 00 X(2) = .13070889378E 01

INTERMEDIATE VALUES 15 GOOD STEPS TAKEN , 0 BAD STEPS TAKEN

T = .6000000000E 01 FPT = .1000000000E 01 DTP = .1000000000E 01 DTG = .1000000000E 00 TOL = .1000000000E-10  
X(1) = .12790420293E 01 X(2) = .24378144489E 01

INTERMEDIATE VALUES 19 GOOD STEPS TAKEN , 0 BAD STEPS TAKEN

T = .7000000000E 01 FPT = .1000000000E 01 DTP = .1000000000E 01 DTG = .1000000000E 00 TOL = .1000000000E-10  
X(1) = .19201524166E 01 X(2) = -.43583853332E 00

TABLE A-4. (Continued)

INTERMEDIATE VALUES      10 GOOD STEPS TAKEN ,      0 BAD STEPS TAKEN  
 T    . .80000000000E 01    FPT    .10000000000E 01    DTP    .10000000000E 01    DTG    .10000000000E 00    TOL    .10000000000E-10  
 X(1)    .12132324410E 01    X(2)    -.98781392226E 00

INTERMEDIATE VALUES      11 GOOD STEPS TAKEN ,      1 BAD STEPS TAKEN  
 T    . .90000000000E 01    FPT    .10000000000E 01    DTP    .10000000000E 01    DTG    .10000000000E 00    TOL    .10000000000E-10  
 X(1)    -.41291605205E 00    X(2)    -.25269034480E 01

INTERMEDIATE VALUES      17 GOOD STEPS TAKEN ,      0 BAD STEPS TAKEN  
 T    . .10000000000E 02    FPT    .10000000000E 01    DTP    .10000000000E 01    DTG    .10000000000E 00    TOL    .10000000000E-10  
 X(1)    -.20083407830E 01    X(2)    .32907070692E-01

INTERMEDIATE VALUES      13 GOOD STEPS TAKEN ,      1 BAD STEPS TAKEN  
 T    . .11000000000E 02    FPT    .10000000000E 01    DTP    .10000000000E 01    DTG    .10000000000E 00    TOL    .10000000000E-10  
 X(1)    -.15049739797E 01    X(2)    .78444442429E 00

INTERMEDIATE VALUES      13 GOOD STEPS TAKEN ,      0 BAD STEPS TAKEN  
 T    . .12000000000E 02    FPT    .10000000000E 01    DTP    .10000000000E 01    DTG    .10000000000E 00    TOL    .10000000000E-10  
 X(1)    -.31376909536E 00    X(2)    .18440269613E 01

INTERMEDIATE VALUES      17 GOOD STEPS TAKEN ,      0 BAD STEPS TAKEN  
 T    . .13000000000E 02    FPT    .10000000000E 01    DTP    .10000000000E 01    DTG    .10000000000E 00    TOL    .10000000000E-10  
 X(1)    .18717708928E 01    X(2)    .99758656955E 00

INTERMEDIATE VALUES      20 GOOD STEPS TAKEN ,      0 BAD STEPS TAKEN  
 T    . .14000000000E 02    FPT    .10000000000E 01    DTP    .10000000000E 01    DTG    .10000000000E 00    TOL    .10000000000E-10  
 X(1)    .17386417030E 01    X(2)    -.62715364635E 00

INTERMEDIATE VALUES      8 GOOD STEPS TAKEN ,      0 BAD STEPS TAKEN  
 T    . .15000000000E 02    FPT    .10000000000E 01    DTP    .10000000000E 01    DTG    .10000000000E 00    TOL    .10000000000E-10

TABLE A-4. (Concluded)

X(1) = .83043742862E 00 X(2) = -.13133658842E 01

INTERMEDIATE VALUES 14 GOOD STEPS TAKEN , 0 BAD STEPS TAKEN

T = .16000000000E 02 FPT = .10000000000E 01 DTP = .10000000000E 01 DTG = .10000000000E 00 TOL = .10000000000E-10  
X(1) = -.12913899229E 01 X(2) = -.24232638004E 01

INTERMEDIATE VALUES 17 GOOD STEPS TAKEN , 1 BAD STEPS TAKEN

T = .17000000000E 02 FPT = .10000000000E 01 DTP = .10000000000E 01 DTG = .10000000000E 00 TOL = .10000000000E-10  
X(1) = -.19179362020E 01 X(2) = .43962156405E 00

INTERMEDIATE VALUES 10 GOOD STEPS TAKEN , 0 BAD STEPS TAKEN

T = .18000000000E 02 FPT = .10000000000E 01 DTP = .10000000000E 01 DTG = .10000000000E 00 TOL = .10000000000E-10  
X(1) = -.12082152063E 01 X(2) = .99161534742E 00

INTERMEDIATE VALUES 10 GOOD STEPS TAKEN , 0 BAD STEPS TAKEN

T = .19000000000E 02 FPT = .10000000000E 01 DTP = .10000000000E 01 DTG = .10000000000E 00 TOL = .10000000000E-10  
X(1) = .42574876675E 00 X(2) = .25353542246E 01

FINAL VALUES 276 GOOD STEPS TAKEN 6 BAD STEPS TAKEN

T = .20000000000E 02 FPT = .10000000000E 01 DTP = .10000000000E 01 DTG = .10000000000E 00 TOL = .10000000000E-10  
X(1) = .20081497614E 01 X(2) = -.42508886890E-01

## APPENDIX B

### A LISTING OF THE RATIONAL FUNCTION EXTRAPOLATION ALGORITHM

A listing of both the old and new versions of DIFSYF is included here. These subroutines are versions of the rational function extrapolation technique as developed by Bulirsch and Stoer [5]. The listings are included here so that they may be compared with other versions that exist.

## Old Version of DIFSYP

```
1.      SUBROUTINE DIFSYP (N,F,EFS,H,X,Y)
2.      DIMENSION Y(17), DTT(17), D(6), YA(17), YL(17), YM(17),
3.      DY(17), D1(17), DT(17,6), YG(8,17), YH(8,17), S(17)
4.      INTEGER R,SR
5.      LOGICAL KONV,BO,BH,FIN
6.      EP=ABS(EPS)
7.      N1=N
8.      HH=H
9.      IF (EP.LT.5.E-8) EP=5.E-8
10.     CALL F (X,Y,DZ)
11.     BH=.FALSE.
12.     FIN=.FALSE.
13.     DO 1 I=1,N1
14.       S(I)=0.
15.     1  YA(I)=Y(I)
16.     2  A=HH+X
17.       FC=1.5
18.       BO=.FALSE.
19.       M=1
20.       R=2
21.       SR=3
22.       JJ=0
23.       DO 23 J=1,10
24.         IF (BO) GO TO 3
25.         D(1)=2.25
26.         D(3)=9.
27.         D(5)=36.
28.         GO TO 4
29.       3  D(1)=1.7777777778
30.         D(3)=7.1111111111
31.         D(5)=28.4444444444
32.       4  KONV=J.GT.5
33.         IF (J.LE.7) GO TO 5
34.         L=6
35.         D(6)=64.
36.         FC=.6*FC
37.         GO TO 7
38.       5  L=J-1
39.         IF (J-1) 7,7,6
40.       6  D(L)=M*M
41.       7  M=M+M
42.       G=HH/FLOAT(M)
```

```

43.      B=G+G
44.      IF (BH.AND.J.LT.9) GO TO 14
45.      KK=(M-2)/2
46.      M=M-1
47.      DO 8 I=1,N1
48.      YL(I)=YA(I)
49.      8  YM(I)=G*DZ(I)+YA(I)
50.      DO 13 K=1,M
51.      CALL F (X+FLOAT(K)*G,YM,DY)
52.      IF (DY(1).GT.1.E38) GO TO 25
53.      DO 10 I=1,N1
54.      U=B*DY(I)+YL(I)
55.      YL(I)=YM(I)
56.      YM(I)=U
57.      U=ABS(U)
58.      IF (U-S(I)) 10,10,9
59.      9  S(I)=U
60.      10 CONTINUE
61.      IF (K.EQ.KK.AND.K.NE.2) GO TO 11
62.      GO TO 13
63.      11 JJ=1+JJ
64.      DO 12 I=1,N1
65.      YH(JJ,I)=YM(I)
66.      12 YG(JJ,I)=YL(I)
67.      13 CONTINUE
68.      GO TO 16
69.      14 DO 15 I=1,N1
70.      YM(I)=YH(J,I)
71.      15 YL(I)=YG(J,I)
72.      16 CALL F (A,YM,DY)
73.      IF (DY(1).GT.1.E38) GO TO 25
74.      DO 22 I=1,N1
75.      V=DTT(I)
76.      DTT(I)=(YM(I)+YL(I)+G*DY(I))* .5
77.      C=DTT(I)
78.      TA=C
79.      IF (L.LT.1) GO TO 20
80.      DO 19 K=1,L
81.      B1=D(K)*V
82.      B=B1-C
83.      U=V

```



```

84.      IF (ABS(B) -1. E-10) 18,18,17
85.      17  B=(C-V)/B
86.      U=C*B
87.      C=B1*B
88.      18  V=DT(I,K)
89.      DT(I,K)=U
90.      19  TA=U+TA
91.      20  IF (ABS(Y(I)-TA).GT.EP*S(I)) GO TO 21
92.      GO TO 22
93.      21  KONV=.FALSE.
94.      22  Y(I)=TA
95.      IF (KONV) GO TO 24
96.      D(2)=4.
97.      D(4)=16.
98.      BO=.NOT.BO
99.      M=R
100.     R=SR
101.     23  SR=M+M
102.     BH=.NOT.BH
103.     FIN=.TRUE.
104.     HH=HH*.5
105.     GO TO 2
106.     24  H=FC*HH
107.     X=A
108.     RETURN
109.     25  H=0
110.     DO 26 I=1,N1
111.     26  Y(I)=YA(I)
112.     END

```

END OF COMPILATION:

NO DIAGNOSTICS.

## New Version of DIFSYF

```

PROTOKOLL
SUBROUTINE DIFSYS(N,F,EPS,H,X,Y)
  EXTERNAL F
  DIMENSION Y(17),YA(21),YL(21),YM(21),DY(21),
  1DZ(21),DT(21,7),D(7),S(21),EP(4)
  LOGICAL KO'IV,BI),GR,KL
  DATA EP/0.4E-1,0.16E-2,0.64E-4,0.256E-5/
  JTI=0
  FY=1.
  ETA=ABS(EPS)
  IF(ETA.LT.1.E-9) ETA=1.E-9
  DO 100 I=1,N
100  YA(I)=Y(I)
  CALL F(X,Y,DZ)
  10  X'V=X+H
  BU=.FALSE.
  DO 110 I=1,N
110  S(I)=C.
  A=1
  JR=2
  JS=3
  DO 200 J=1,10
  IF(.NOT.BU) GO TO 200
  D(2)=1.77777777778
  D(4)=7.11111111111
  D(6)=28.4444444444
  GO TO 201
200  D(2)=2.25
  D(4)=9.
  D(6)=36.
201  IF(J.LF.7) GO TO 202
  L=7
  D(7)=64.
  GO TO 203
202  L=J
  D(L)=M*M
203  KO'IV=L.GT.3
  M=M+4
  G=H/FLD'AT(M)
  B=G+G
  DO 210 I=1,M
  YL(I)=YA(I)
210  YM(I)=YA(I)+G*DZ(I)
  M=M+1
  DO 220 K=1,M
  CALL F(X+FLD'AT(K)*G,YM,DY)
  DO 220 I=1,N
  U=YL(I)+B*DY(I)
  YL(I)=YM(I)
  YM(I)=L
  U=ABS(L)
220  IF(U.GT.S(I)) S(I)=U
  CALL F(X,YM,DY)
  KL=L.LT.2
  GR=L.GT.5
  FS=0.
  DO 233 I=1,N
  V=DT(I,1)
  C=(YM(I)+YL(I)+G*DY(I))*0.5
  DT(I,1)=C
  TA=C
  IF(KL) GO TO 233

```

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```

      DD 231 K=2,L
      B1=D(K)*V
      D=B1-C
      H=C-V
      I=V
      IF(B.EQ.0. ) GOTO 230
      B=B/3
      U=C*B
      C=B1*B
230  V=DT(I,K)
      DT(I,K)=U
231  TA=U+TA
      IF(.NOT.KONV) GOTO 232
      IF(ABS(Y(I)-TA).GT.S(I)*ETA) KONV=.FALSE.
232  IF(GR.CR.S(I).EQ.0.) GO TO 233
      FV=ABS(W)/S(I)
      IF(FS.LT.FV) FS=FV
233  Y(I)=TA
      IF(FS.EQ.0.) GOTO 250
      FA=FY
      K=L-1
      FY=(EP(F)/FS)**(1./FLOAT(L+K))
      IF(L.EQ.2) GOTO 240
      IF(FY.LT.0.7*FA) GO TO 250
24  IF(FY.GT.0.7) GOTO 250
      H=H*FY
      JTI=JTI+1
      IF(JTI.GT. 5) GO TO 30
      GO TO 10
250 IF(KONV) GO TO 20
      D(3)=4.
      D(5)=16.
      BU=.NOT.BU
      H=JR
      JR=JS
260 JS=M+M
      H=H*.5
      GO TO 10
20  X=X1
      H=H*FY
      RETURN
30  H=0.
      DD 300 I=1,I
300 Y(I)=YA(I)
      RETURN
      END

```

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A COMPARISON OF DIGITAL COMPUTER PROGRAMS FOR THE  
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DIFFERENTIAL EQUATIONS

By Hugo L. Ingram

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